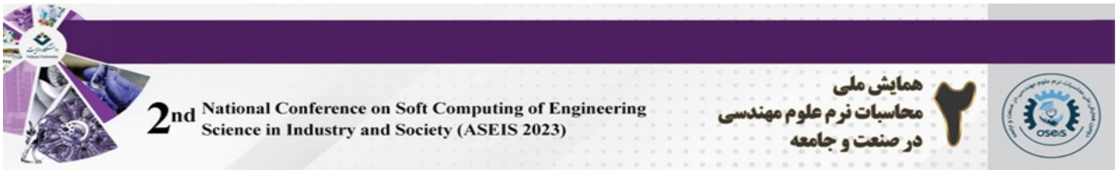

Pervin Nominal Spaces

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Abstract. Nominal set theory provides a mathematical framework for studying semantics, modifying variables, and much more in computer science. Each nominal set X can be equipped with a preorder relation that leads us to various topologies on X. Here we using this preorder and introduce pervin space associated with each nominal set so-called pervin nominal space. We also examine some topological properties of pervin nominal spaces such as separation axioms and compactness.

Keywords: support, nominal set, pervin space

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1. Introduction

Nominal set theory provides a mathematical framework for studying semantics, modifying variables, and much more in computer science. Fraenkel presented nominal sets in [3] as an alternative model of set theory in 1922. In this context Mostowski studied further, which is why nominal sets are sometimes referred to as Fraenkel-Mostowski sets. In the 1990s, Gabbay and Pitts [6] rediscovered nominal sets for the computer science community, and this notion sparked a lot of interest in semantics [1, 2, 4, 5].

It is shown, in [7], that a nominal set is equipped with support-preordered and using this preorder authors introduce nominal spaces. Here we using this preorder and introduce pervin space associated with each nominal set so-called pervin nominal space. We also examine some topological properties of pervin nominal spaces such as separation axioms and compactness.

Now we give some necessary notions on nominal sets needed throughout the paper from [9]. For category theory information one may consult [1].

Given a set $\mathbb{D}$, a permutation π of $\mathbb{D}$ is a bijective map from $\mathbb{D}$ to itself. The permutations of $\mathbb{D}$ with composition and identity form a group, called the *symmetric group* on the set $\mathbb{D}$ and denoted by $\mathrm{Sym}(\mathbb{D})$. A permutation $\pi \in \mathrm{Sym}(\mathbb{D})$ is finitary if the set

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$\{d \in \mathbb{D} \,|\, \pi d \neq d\}$ is a finite subset of $\mathbb{D}$. It is clear that $\mathrm{id} \in \mathrm{Sym}(\mathbb{D})$ is finitary and that the composition and the inverse of finitary permutations are finitary. Therefore, we get a subgroup of $\mathrm{Sym}(\mathbb{D})$ of finitary permutations, denoted by $\mathrm{Perm}(\mathbb{D})$. We fix a countable infinite set $\mathbb{D}$ whose elements are denoted by $a, b, c, \ldots$ and called *atomic names*.

Let X be a set equipped with an action of the group $\mathrm{Perm}(\mathbb{D})$, $\mathrm{Perm}(\mathbb{D}) \times X \to X$ mapping (π, x) to πx. We call X a $\mathrm{Perm}(\mathbb{D})$-*set*, whenever for every $\pi_1, \pi_2 \in \mathrm{Perm}(\mathbb{D})$ and every $x \in X$ we have:

(1) $\pi_1(\pi_2 x) = (\pi_1 o \pi_2)x$
(2) $\mathrm{id}\, x = x$.

A subset Y of a $\mathrm{Perm}(\mathbb{D})$-set X is called *equivariant* if $\pi y \in Y$, for all $\pi \in \mathrm{Perm}(\mathbb{D})$ and $y \in Y$. $\mathrm{Perm}(\mathbb{D})$-sets are the objects of a category, denoted by $\mathrm{Perm}(\mathbb{D})$-**Set** whose morphisms are equivariant maps, i.e. maps subject to the rule $f(\pi x) = \pi f(x)$, for all $x \in X, \pi \in \mathrm{Perm}(\mathbb{D})$, whose compositions and identities are as in the category **Set** of sets and maps.

An element x of a $\mathrm{Perm}(\mathbb{D})$-set X is called a *zero* element if $\pi x = x$, for all $\pi \in \mathrm{Perm}(\mathbb{D})$. The set of all zero elements of the $\mathrm{Perm}(\mathbb{D})$-set X is denoted by $\mathcal{Z}(X)$. A $\mathrm{Perm}(\mathbb{D})$-set all of whose elements are zero is called *discrete*.

Given a $\mathrm{Perm}(\mathbb{D})$-set X, a set of atomic names $C \subseteq \mathbb{D}$ is a *support* for an element $x \in X$ if for all $\pi \in \mathrm{Perm}(\mathbb{D})$ with $\pi(d) = d$, we have $\pi x = x$. Given a $\mathrm{Perm}(\mathbb{D})$-set X, we say an element $x \in X$ is finitely supported, if there is some finite set of atomic names that is, a support for the element x.

A *nominal set* is a $\mathrm{Perm}(\mathbb{D})$-set all of whose elements are finitely supported. Nominal sets are the objects of a category, denoted by **Nom**, whose morphisms are equivariant maps and whose compositions and identities are as in the category of $\mathrm{Perm}(\mathbb{D})$-**Set**.

Remark 1.1. Suppose X is a nominal set and $x \in X$. Intersection of two supports of x is a (finite) support of x, [**9**, Propositions 2.1 and 2.3]. So each $x \in X$ has the least (finite) support which is denoted by $\mathrm{supp}_X\, x$, and when there is no possibility of error, we denote it by $\mathrm{supp}\, x$. In fact, $\mathrm{supp}\, x = \bigcap\{C : C \text{ is a finite support of } x\}$.

Given a nominal set X, a subset $Y \subseteq X$ is called *uniformly supported* if there exists a finite set $C \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})$ that supports each $y \in Y$. Notice that, in this case, $\pi Y = Y$, for all $\pi \in \mathrm{Perm}(\mathbb{D})$ with $\pi d = d$ for every $d \in C$. So, $Y \in \mathcal{P}_{\mathrm{fs}}(X)$.

A set of atomic names $A \subseteq \mathbb{D}$ is called *strongly supports* an element x of a nominal set X if and only if

$$(\forall \pi \in \mathrm{Perm}(\mathbb{D}))(\forall a \in A,\ \pi(a) = a) \Leftrightarrow \pi x = x.$$

A strong nominal set is a $\mathrm{Perm}(\mathbb{D})$-set in which every element is strongly supported by a finite set of atomic names.

Example 1.2. The nominal set $\mathcal{P}_{\mathrm{f}}(\mathbb{D})$ is not strongly nominal set. Because for some $d \neq d' \in \mathbb{D}$ we have $(d, d')\{d, d'\} = \{d, d'\}$, but $(d, d')(d) = d' \neq d$.

In the following theorem we characterize the structure of strong nominal sets.

Theorem 1.3. *The nominal set X is a strong nominal set if and only if X is isomorphic to an equivarian subset of $\dot{\bigcup}_{n \in \mathbb{N}^0} \mathbb{D}^{(n)}$.*

With the following proposition in mind, we give the next Theorem.

Proposition 1.4. *[**9**] Suppose X is a $\mathrm{Perm}(\mathbb{D})$-set and $x \in X$. A subset $A \subseteq \mathbb{D}$ supports x if and only if, for all $d_1, d_2 \in \mathbb{D} \setminus A$, we have $(d_1\ d_2)x = x$.*

Theorem 1.5. *Suppose X is a nominal set, $x \in X$ and A is a finite subset of $\mathbb{D}$. Then, the following statements are equivalent:*

(i) $\operatorname{supp} x = A$.

(ii) *For all $d_1, d_2 \in \mathbb{D} \setminus A$, we have $(d_1\ d_2)x = x$ and for all $d_1 \in \mathbb{D}$ and $d_2 \in \mathbb{D} \setminus A$, we have $(d_1\ d_2)x \neq x$.*

Remark 1.6. [9, Theorem 5.1] Every finite nominal set is discrete.

2. The set $\mathcal{L}_X$ and its properties

We recall from [7] that by the *support-preorder* on a nominal set X, we mean the binary relation $\preceq$ on X defined by:

$$x \preceq y \Leftrightarrow \operatorname{supp} x \subseteq \operatorname{supp} y.$$

Since $\preceq$ is a preorder (i.e. reflexive and transitive), the pair $(X, \preceq)$ is called a *support-preordered nominal set* or briefly *sp-nominal set*. It can be easily seen that the support-preorder is equivariant (or action preserving); meaning that:

$$x_1 \preceq x_2 \Rightarrow \pi x_1 \preceq \pi x_2,$$

for each $x_1, x_2 \in X, \pi \in \operatorname{Perm}(\mathbb{D})$. An equivariant map $f : X \to Y$ is called *support-preorder preserving* (or for convenience *sp-preserving*) whenever $f(x_1) \preceq f(x_2)$, for all $x_1 \preceq x_2 \in X$.

Lemma 2.1. *[7, Lemma 3.4] Let X be an sp-nominal set and $x, x' \in X$. Then, there exists π with $\pi x \preceq x'$ or $\pi x' \preceq x$.*

Lemma 2.2. *Let X is a nominal set and x is a non-zero element of X. Then, for each $\{d_1, d_2, \ldots, d_k\} \subseteq \operatorname{supp} x$, there exists $\pi \in \operatorname{Perm}(\mathbb{D})$ such that $\operatorname{supp} \pi x \cap \operatorname{supp} x = \{d_1, d_2, \ldots, d_k\}$.*

Lemma 2.3. *[7, Lemma 3.13] Suppose X and Y are two sp-nominal sets, $f : X \to Y$ is an sp-preserving map, and $x \in X$ with $\operatorname{supp} f(x) \neq \emptyset$. Then, $\operatorname{supp} f(x) = \operatorname{supp} x$.*

Definition 2.4. Given a nominal set X and $x \in X$, and $L \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})$, we define:

$$X^{(L)} \triangleq \{x \in X : \ \operatorname{supp} x \subseteq L\}.$$

Lemma 2.5. *Let X be a nominal set and $L \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})$. Then,*

(i) *$X^{(L)}$ is a finitely supported subset of X.*

(ii) *Y is a uniformly supported subset of X if and only if there exists a finite $L \subseteq \mathbb{D}$ with $Y \subseteq X^{(L)}$.*

(iii) $X^{(L)} = \bigcup_{\operatorname{supp} x \subseteq L} X^{(\operatorname{supp} x)}$

Corollary 2.6. (i) *For a given family $\{K_\alpha\} \subseteq \mathcal{P}_{\mathrm{f}}(\mathbb{D})$ and a nominal set X, we have $\bigcap_{\alpha \in I} X^{(K_\alpha)} = X^{\bigcap_\alpha (K_\alpha)}$.*

(ii)

$$X^{(L)} \cup (X^{(K)} \cap X^{(P)}) = X^{(L)} \cup X^{(K \cap P)}$$
$$\subseteq X^{(L \cup (K \cap P))} = X^{((L \cup K) \cap (L \cup P))} = X^{(L \cup K)} \cap X^{(L \cup P)}.$$

(iii) $X^{(L)} \cap (X^{(K)} \cup X^{(P)}) \subseteq X^{(L)} \cap X^{(K \cup P)} = X^{(L \cap (K \cup P))}$.

Remark 2.7. Take X to be a nominal set and $L \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})$. Then, $\mathcal{L}_X = \{X^{(L)} : L \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})\}$ is a an up-directed nominal set.

Theorem 2.8. *Let X be a nominal set and $L \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})$. Then $\mathcal{L}_X = \{X^{(\mathrm{supp}\, x)} : x \in X\}$.*

Remark 2.9. Let $f : X \to Y$ be an equivariant map and $L \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})$. Then,
(i) $X^{(L)} \subseteq f^{-1}(Y^{(L)})$.
(ii) $f(X^{(L)}) \subseteq (f(X))^{(L)}$.
(iii) if $\mathcal{Z}(Y) = \emptyset$ then, $X^{(L)} = f^{-1}(Y^{(L)})$ and $f(X^{(L)}) = (f(X))^{(L)}$.
(iv) if $\mathcal{Z}(Y) = \emptyset$ and f is surjective, then $f(X^{(L)}) = Y^{(L)}$. It is always true $f(X^{(L)}) \subseteq Y^{(L)}$.
(v) L supports $f(X^{(L)})$ and $f^{-1}(Y^{(L)})$.
(vi) $f(X^{(L)}) = Y^{(L)}$ if and only if for each $y \in Y^{(L)}$ there exists $x \in X^{(L)}$ with $y = f(x)$.

3. Nominal Pervin Spaces

Using Remark 2.7, one can consider the topology

$$\tau_X = \{U \in \mathcal{P}_{\mathrm{fs}}(X) : U = \bigcup_{\{A_i\}_{i\in I} \subseteq \mathcal{L}_X} A_i\} \cup \{\emptyset\}$$

on X generated by the basis $\mathcal{L}_X$. A nominal set together with this topology, (X, τ_X) is called a *nominal pervin space*. By Theorem 2.8, τ_X arises from $\{X^{(\mathrm{supp}\, x)} : x \in X\}$ as the subbasis.

Remark 3.1. Let X be a nominal pervin space. Then, for every $x \in X$, $X^{(\mathrm{supp}\, x)}$ is the smallest open set contains x.

Proof. It is clear, since $X^{(\mathrm{supp}\, x)} \in \mathcal{L}_X$ contains x, for every $x \in X$, and the topology is generated by $\mathcal{L}_X$. □

Corollary 3.2. *Given a nominal set X, we have:*
(i) $X = \bigcup_{x\in X} X^{(\mathrm{supp}\, x)}$
(ii) *if* $X^{(\mathrm{supp}\, x)} \cap X^{(\mathrm{supp}\, x')} \neq \emptyset$, *then* $X^{(\mathrm{supp}\, x)} = X^{(\mathrm{supp}\, x')}$.
(iii) *for each* $x \in X$, $X^{(\mathrm{supp}\, x)} \neq \emptyset$.
(iv) $X^{\emptyset} = \mathcal{Z}(X)$.

Remark 3.3. Let X be a nominal set. Then one can define the equivalence relation $\sim$ on $\mathcal{L}_X$ as

$$X^{(L)} \sim X^{(L')} \Leftrightarrow |L| = |L'|.$$

The quotient set $\mathcal{L}_X/\sim$ together with the canonical action over $\mathrm{Perm}(\mathbb{D})$, $\pi(X^{(L)}/\sim) = (\pi X^{(L)})/\sim = X^{(\pi L)}/\sim$, is a nominal set.

Theorem 3.4. *Let X be a nominal set. Then $\mathcal{L}_X/\sim$ is isomorphic to subset of* $\mathbb{N}^0$.

Proof. Considering $f : X/\sim \to \mathbb{N}^0$ defined by $X^{(L)}/\sim \mapsto |L|$ and $\{|L| : L \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})\}$ with the discrete action. We show f is well-defined. Since if $X^{(L)}/\sim = X^{(L')}/\sim$. Hence, $X^{(L)} \sim X^{(L')}$. So, $|L| = |L'|$. Also,

$$f(\pi X^{(L)}/\sim) = f(X^{(\pi L)})/\sim = |\pi L| = |L| = \pi f(X^{(L)}/\sim)$$

for each $\pi \in \mathrm{Perm}(\mathbb{D})$. Hence, f is equivariant map. Since,

$$\begin{aligned} \ker f &= \{(X^{(L_1)}/\sim, X^{(L_2)}/\sim) : f(X^{(L_1)}/\sim) = f(X^{(L_2)}/\sim)\} \\ &= \{(X^{(L_1)}/\sim, X^{(L_2)}/\sim) : |L_1| = |L_2|\} = \Delta. \end{aligned}$$

□

The following example shows that every closed set is not necessarily open in a pervin nominal space.

EXAMPLE 3.5. Suppose $X = \mathbb{D} \cup \{\theta\}$. Take $d \in \mathbb{D}$. Considering the closed set $F = X \setminus X^{(\{d\})}$. If F is an open set, then $X^{(\{d_1\})} \subseteq F$ for $d_1 \neq d \in \mathbb{D}$. Hence, $\theta \in F$ which is a contradiction. Because $\theta \in X^{(\{d\})}$. Therefore, F is not open.

It is worth noting that an equivariant map between nominal sets does not necessarily sp-preserving, see [7, Examples 3.6 and 3.7]. Now we give the following statements concerning sp-preserving maps

LEMMA 3.6. *Let $f : X \to Y$ be a sp-preserving map between nominal sets. Then, for all $L \in \mathcal{P}_{\mathrm{f}}(\mathbb{D})$, $f^{-1}(Y^{(L)}) \in \tau_X$.*

THEOREM 3.7. *Let $f : X \to Y$ be an equivariant map between nominal sets. Then $f : X \to Y$ is sp-preserving if and only if f is continuous.*

In the sequel of this section we examine separation axioms and describe compact pervin nominal spaces.

THEOREM 3.8. *The pervin nominal space X is T_0 if and only if the support map* $\mathrm{supp} : X \to \mathcal{P}_{\mathrm{f}}(\mathbb{D})$ *separates distinc members of X i.e. for each $x_1 \neq x_2 \in X$,* $\mathrm{supp}\, x_1 \neq \mathrm{supp}\, x_2$.

THEOREM 3.9. *Let (X, τ_X) be a pervin nominal space and F is a closed subset of X. Then,*

$$X \setminus \bigcup_{x \in X \setminus F} X^{(\mathrm{supp}\, x)} = \bigcup_{x \in X \setminus F} (X \setminus X^{(\mathrm{supp}\, x)}).$$

THEOREM 3.10. *Let (X, τ_X) be a pervin nominal space. Then, X is a regular space if and only if for each $x \in X$, $|\mathrm{supp}\, x| = n$ for some $n \in \mathbb{N}^0$.*

THEOREM 3.11. *Let (X, τ_X) be a pervin nominal space. Then, X is a normal space if and only if $\{|\mathrm{supp}\, x| : x \in X\}$ of $\mathbb{N}^0$ has no upper bound or for each $x \in X$, $|\mathrm{supp}\, x| = n$ for some $n \in \mathbb{N}^0$.*

THEOREM 3.12. *Let (X, τ_X) be a pervin nominal space. Then, the following statements are equivalent:*
(i) *X is T_1.*
(ii) *$X \cong \{A \in \mathcal{P}_{\mathrm{f}}(\mathbb{D}) : |A| = n\}$ for some $n \in \mathbb{N}^0$.*
(iii) *X is T_2.*
(iv) *X is a separator for each $A, B \in \mathcal{P}(X)$ with $A \cap B = \emptyset$.*
(v) *The relation $\preceq$ is a partially order on X.*

THEOREM 3.13. *Let X be a pervin nominal space. The following statement are equivalent:*

(i) *X is compact.*
(ii) *X is a discrete nominal set.*
(iii) *$\tau_X = \{\emptyset, X\}$.*

4. Conclusion

Here we show the relationship between algebraic properties in a nominal set and topological properties which is defined by support in the nominal set. We examine some of its topological properties, in particular separation axioms for example, in Theorem 3.8, we specify T_0 pervin spaces. Also, we show that continuous maps are sp-preserving maps.

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