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Hilbert-Schmidt Dual Frames

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Abstract. A Hilbert–Schmidt frame is a more general version of the g-frame, an important generalization of ordinary frames. In this paper, we discuss the dual of Hilbert–Schmidt frames in separable Hilbert spaces and establish the links between dual of Hilbert–Schmidt frames and dual of ordinary frames.

Keywords: Hilbert-Schmidt frames, dual frames, g-frames.

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1. Introduction

The von Neumann–Schatten frames in a separable Banach space was first proposed by Sadeghi and Arefijamaal [20] to deal with all the existing frames as a united object. In fact, von Neumann-Schatten frames is an extension of g-frames [21], bounded quasi-projectors [12], fusion frames [3, 5], pseudo-frames [14], weighted frames [4], oblique frames [7, 11], outer frames [1], p-frames for separable Banach spaces [9] and in the context of numerical analysis the stable space splittings [15, 16]. As an important class of von Neumann-Schatten p-frames, Hilbert-Schmidt frames have interested some mathematicians due to having the inner product structure [8,9]. For more information on Hilbert-Schmidt frames, see Refs. [13, 17, 18, 22].

Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\mathcal{E} = \{e_n\}_{n\in\mathbb{N}}$ and $B(\mathcal{H})$ denotes the C^*-algebra of all bounded linear operators on $\mathcal{H}$. For a compact operator $\mathcal{A} \in B(\mathcal{H})$, let $s_1(\mathcal{A}) \geq s_2(\mathcal{A}) \geq \cdots \geq 0$ denote the singular values of $\mathcal{A}$, that is, the eigenvalues of the positive operator $|\mathcal{A}| = (\mathcal{A}^*\mathcal{A})^{\frac{1}{2}}$, arranged in a decreasing order and repeated according to multiplicity. The Hilbert-Schmidt class $\mathcal{C}_2(\mathcal{H})$ ($\mathcal{C}_2 := \mathcal{C}_2(\mathcal{H})$ for simplicity) is defined to be the set of all compact operators A for which $\sum_{i=1}^{\infty} s_i^2(\mathcal{A}) < \infty$. For $\mathcal{A} \in \mathcal{C}_2$, the Hilbert-Schmidt norm of $\mathcal{A}$ is defined by

$$\|\mathcal{A}\|_{\mathcal{HS}} = \Big(\sum_{i=1}^{\infty} s_i^2(\mathcal{A})\Big)^{\frac{1}{2}} = \Big(\mathbf{tr}|\mathcal{A}|^2\Big)^{\frac{1}{2}}, \tag{1}$$

where $\mathbf{tr}$ is the trace functional which defines as $\mathbf{tr}(\mathcal{A}) = \sum_{n\in\mathbb{N}} \langle \mathcal{A}(e_n), e_n\rangle$. It is shown that the space $\mathcal{C}_2$ with the inner product $[T, S]_{\mathbf{tr}} := \mathbf{tr}(S^*T)$ is a Hilbert space.

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Now, we define the Hilbert space

$$\oplus\mathcal{C}_2 = \Big\{\mathcal{A} = \{\mathcal{A}_i\}_{i=1}^{\infty} : \ \mathcal{A}_i \in \mathcal{C}_2 \quad \forall i \in \mathbb{N} \text{ and } \|\mathcal{A}\|_2 := \Big(\sum_i \|\mathcal{A}_i\|_{\mathcal{HS}}^2\Big)^{\frac{1}{2}} < \infty\Big\}.$$

with the inner product $\langle \mathcal{A}, \mathcal{A}' \rangle := \sum_{i=1}^{\infty} [\mathcal{A}_i, \mathcal{A}_i']_{\mathbf{tr}}$, and so $\|\mathcal{A}\|_2^2 = \langle \mathcal{A}, \mathcal{A} \rangle$([10]).

If x and y are elements of a Hilbert spaces $\mathcal{H}$ we define the operator $x \otimes y$ on $\mathcal{H}$ by

$$(x \otimes y)(z) = \langle z, y \rangle\, x.$$

It is obvious that $\|x \otimes y\| = \|x\|\|y\|$ and the rank of $x \otimes y$ is one if x and y are non-zero. If $x, x', y, y' \in \mathcal{H}$ and $u \in B(\mathcal{H})$, then the following equalities are easily verified:

$$\begin{aligned}
(x \otimes x')\,(y \otimes y') &= \langle y, x' \rangle\,(x \otimes y') \\
(x \otimes y)^* &= y \otimes x \\
u\,(x \otimes y) &= u(x) \otimes y \\
(x \otimes y)\,u &= x \otimes u^*(y).
\end{aligned}$$

Note that if $x, y \in \mathcal{H}$, then $\|x \otimes y\|_{\mathcal{HS}} = \|x\|\|y\|$ and $\mathbf{tr}(x \otimes y) = \langle x, y \rangle$ so $x \otimes y$ is in $\mathcal{C}_2$. The operator $x \otimes x$ is a rank-one projection if and only if $\langle x, x \rangle = 1$, that is, x is a unit vector. Conversely, every rank-one projection is of the form $x \otimes x$ for some unit vector x. If $\{\eta_i : i \in I\}$ and $\{\zeta_i : i \in I\}$ are orthonormal bases in $\mathcal{H}$, then $\{\eta_i \otimes \zeta_j : i, j \in I\}$ is an orthonormal basis of $\mathcal{C}_2$; see [19] for more details.

Definition 1.1 ([6]). A sequence $\{f_i\}_{i\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist positive constants A, B such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |< f, f_i >|^2 \leq B\|f\|^2 \tag{2}$$

for all $f \in \mathcal{H}$. It is called a Parseval frame if $A = B = 1$ in (2). It is called a Bessel sequence with bound B if the right-hand side inequality of (2) holds.

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces, $B(\mathcal{K}, \mathcal{H})$ be the set of all bounded linear operators from $\mathcal{K}$ to $\mathcal{H}$ and $\{\mathcal{H}_i : i \in \mathbb{N}\}$ be a sequence of closed subspaces of $\mathcal{H}$.

Definition 1.2 ([21]). A sequence $\{\Lambda_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in \mathbb{N}\}$ is called a generalized frame or simply a g-frame for $\mathcal{K}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{N}\}$ if there exist constants positive constants A, B such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} \|\Lambda_i(f)\|^2 \leq B\|f\|^2$$

for all $f \in \mathcal{K}$.

Example 1.3 ([21]). Let $\{f_i\}_{i\in\mathbb{N}}$ be a frame for $\mathcal{K}$. Let Λ_i be the functional induced by f_i , i.e.,

$$\Lambda_i(f) =< f, f_i >, \quad i \in \mathbb{N},\ f \in \mathcal{K}.$$

Then $\{\Lambda_i \in B(\mathcal{K}, \mathbb{C}) : i \in \mathbb{N}\}$ is a g-frame for $\mathcal{K}$ with respect to $\mathbb{C}$. By the Riesz Representation Theorem, to every functional $\Lambda \in B(\mathcal{K}, \mathbb{C})$, one can find some $z \in \mathcal{K}$ such that $\Lambda(f) =< f, z >$ for all $f \in \mathcal{K}$. Hence a frame is equivalent to a g-frame whenever $\mathcal{H}_i = \mathbb{C}$, $i \in \mathbb{N}$.

Definition 1.4 ([20]). A sequence $\mathcal{G} := \{\mathcal{G}_i\}_{i=1}^{\infty} \subseteq B(\mathcal{K}, \oplus\mathcal{C}_2)$ is said to be a *Hilbert–Schmidt frame* or simply a HS-*frame* for Hilbert space $\mathcal{K}$ with respect to Hilbert space $\mathcal{H}$, whenever there exist two positive numbers $A_{\mathcal{G}}$ and $B_{\mathcal{G}}$ such that

$$A_{\mathcal{G}}\|f\|^2 \leq \sum_{i=1}^{\infty} \|\mathcal{G}_i(f)\|^2_{\mathcal{HS}} \leq B_{\mathcal{G}}\|f\|^2 \tag{3}$$

for all $f \in \mathcal{K}$. The constants $A_{\mathcal{G}}$ and $B_{\mathcal{G}}$ are called the lower and upper HS-frame bounds of $\mathcal{G}$ and $\mathcal{G}$ is called to be a HS-Bessel sequence for $\mathcal{K}$ with respect to $\mathcal{H}$, if the right-hand side of (3) holds.

Example 1.5 ([20]). Let $\{\Lambda_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in \mathbb{N}\}$ be a g-frame for $\mathcal{K}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{N}\}$. Then $\{\Lambda_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in \mathbb{N}\}$ is a HS-frame for $\mathcal{K}$ with respect to Hilbert space $\mathcal{H} = \bigoplus \mathcal{H}_i$.

By using the Hilbert properties of the spaces, they observed that

$$T_{\mathcal{G}}(\{\mathcal{A}_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \mathcal{G}_i^* \mathcal{A}_i \qquad \text{and} \qquad T_{\mathcal{G}}^*(f) = \{\mathcal{G}_i(f)\}_{i=1}^{\infty},$$

where $f \in \mathcal{K}$ and $\{\mathcal{A}_i\}_{i=1}^{\infty} \in \oplus\mathcal{C}_2$ and the mapping

$$S_{\mathcal{G}} : \mathcal{K} \to \mathcal{K} \qquad , \qquad S_{\mathcal{G}}(f) := T_{\mathcal{G}} T_{\mathcal{G}}^*(f) = \sum_{i=1}^{\infty} \mathcal{G}_i^* \mathcal{G}_i(f)$$

is an invertible, self-adjoint, positive and bounded linear operator and

$$A_{\mathcal{G}} Id_{\mathcal{K}} \leq S_{\mathcal{G}} \leq B_{\mathcal{G}} Id_{\mathcal{K}}.$$

From this, they were able to characterize all dual frames of a HS-frame. It is worthwhile to mention that a HS-frame is a more general version of the g-frame, an important generalization of ordinary frames.

2. Main results

In what follows we shall frequently make use of the following notation for a HS-frame $\mathcal{F}$:

$$\texttt{rann}_{B(\mathcal{K},\oplus\mathcal{C}_2)}(T_{\mathcal{F}}) := \Big\{\Phi \in B(\mathcal{K}, \oplus\mathcal{C}_2) : \ T_{\mathcal{F}}\Phi = 0\Big\},$$

the set of all right annihilators of the operator $T_{\mathcal{F}}$ in $B(\mathcal{K}, \oplus\mathcal{C}_2)$.

Definition 2.1 ([2]). Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^{\infty}$ be a HS-frame for $\mathcal{K}$ with respect to $\mathcal{H}$. A HS-frame $\{\mathcal{G}_i\}_{i=1}^{\infty}$ is called Hilbert-Schmidt dual frame or simply a HS-dual frame for $\mathcal{F}$ if $f = \sum_{i=1}^{\infty} \mathcal{F}_i^* \mathcal{G}_i(f)$ for all $f \in \mathcal{K}$, i.e. $T_{\mathcal{F}} T_{\mathcal{G}}^* = Id_{\mathcal{K}}$.

By using the properties of the Hilbert-Schmidt frame operator $S_{\mathcal{F}}$, we observe that for all $f \in \mathcal{K}$

$$f = S_{\mathcal{F}}(S_{\mathcal{F}}^{-1}f) = \sum_{i=1}^{\infty} \mathcal{F}_i^* \mathcal{F}_i S_{\mathcal{F}}^{-1}(f) \text{ and } f = S_{\mathcal{F}}^{-1}(S_{\mathcal{F}}f) = \sum_{i=1}^{\infty} \ S_{\mathcal{F}}^{-1} \mathcal{F}_i^* \mathcal{F}_i(f).$$

The sequence $\widetilde{\mathcal{F}} = \{\widetilde{\mathcal{F}}_i\}_{i=1}^{\infty} := \{\mathcal{F}_i S_{\mathcal{F}}^{-1}\}_{i=1}^{\infty}$ is a HS-dual frame for $\mathcal{K}$ with respect to $\mathcal{H}$ with the lower and upper HS-bounds $B_{\mathcal{F}}^{-1}$ and $A_{\mathcal{F}}^{-1}$, where $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$ are the lower and upper HS-frame bounds of $\mathcal{F}$ (see Ref. [2]). The HS-dual frame $\widetilde{\mathcal{F}}$ is called the canonical HS-dual frame of $\mathcal{F}$. The authors in [2] characterized all HS-duals of $\mathcal{F}$ by using the canonical HS-dual. Indeed, if $\mathcal{F}$ be a HS-frame of $\mathcal{H}$, then $\mathcal{F}^d = \{\mathcal{F}_i^d\}_{i=1}^{\infty}$ is a HS-dual of $\mathcal{F}$ if and only if

$$\mathcal{F}_i^d = \widetilde{\mathcal{F}}_i + \pi_i\Phi = \mathcal{F}_i S_{\mathcal{F}}^{-1} + \pi_i\Phi,$$

where $\Phi \in \mathtt{rann}_{B(\mathcal{K},\oplus\mathcal{C}_2)}(T_{\mathcal{F}})$ and

$$\pi_i : \oplus\mathcal{C}_2 \to \mathcal{C}_2 \quad , \quad \pi_i\left(\{\mathcal{A}_j\}_j\right) = \mathcal{A}_i.$$

In what follows, the notation $\widetilde{\mathcal{F}}(\Phi)$ denote the HS-dual $\{\widetilde{\mathcal{F}}_i + \pi_i\Phi\}_{i=1}^{\infty}$ of $\mathcal{F}$.

Theorem 2.2 ([2]). *Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^{\infty}$ be a HS-frame with an upper bound B. There exists a one-to-one correspondence between the duals of $\mathcal{F}$ and $\mathtt{rann}_{B(\mathcal{K},\oplus\mathcal{C}_2)}(T_{\mathcal{F}})$.*

The largest lower frame bound and the smallest upper frame bound in (3) are called the optimal frame bounds. Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^{\infty}$ be a HS-frame with the optimal frame bounds A and B. It is not difficult to see that $B = \|S_{\mathcal{F}}\|$ and $A = \|S_{\mathcal{F}}^{-1}\|^{-1}$, moreover, $\frac{1}{B}$ and $\frac{1}{A}$ are the optimal frame bounds of $\widetilde{\mathcal{F}} = \{\mathcal{F}_i S_{\mathcal{F}}^{-1}\}_{i=1}^{\infty}$. The relation between optimal bounds of a frame and its duals is discussed in the following theorem

Theorem 2.3 ([2]). *Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^{\infty}$ be a HS-frame with the optimal frame bounds A and B, and $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$ be its dual with the optimal frame bounds C and D. Then $AD \geq 1$ and $BC \geq 1$. Furthermore, the following are equivalent:*
(1) $D = \frac{1}{A}$
(2) $C = \frac{1}{B}$
(3) *$\mathcal{G}$ is the canonical dual.*

Let $\{\mathcal{F}_i\}_{i=1}^{\infty}$ be a HS-frame for $\mathcal{K}$ with respect to $\mathcal{H}$ and $\{\nu_{n,m} : n, m \in \mathbb{N}\}$ be a orthonormal basis of $\mathcal{C}_2$. Define a bounded linear functional on $\mathcal{K}$ as follows

$$f \mapsto [\mathcal{F}_j f,\ \nu_{n,m}]_{\mathbf{tr}} \quad (f \in \mathcal{K}).$$

By Riesz representation theorem, there exists $f_{j,n,m} \in \mathcal{K}$ such that

$$[\mathcal{F}_j f,\ \nu_{n,m}]_{\mathbf{tr}} = \langle f,\ f_{j,n,m}\rangle \quad (f \in \mathcal{K}). \tag{4}$$

Hence

$$\mathcal{F}_j f = \sum_{n,m\in\mathbb{N}} \langle f,\ f_{j,n,m}\rangle \nu_{n,m} \quad (f \in \mathcal{K}).$$

Theorem 2.4 ([2]). *Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^{\infty}$ be a HS-frame and $f_{j,n,m}$ be defined as in (4). The sequence $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^{\infty}$ is a HS-frame for $\mathcal{K}$ with respect to $\mathcal{H}$ if and only if $\{f_{j,n,m} : j, n, m \in \mathbb{N}\}$ is a (ordinary) frame for $\mathcal{K}$.*

Now, we present the main result as follows.

Theorem 2.5. *Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^{\infty}$ and $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$ be HS-frames for $\mathcal{K}$ with respect to $\mathcal{H}$. Let $\{\nu_{n,m} :\ n, m \in \mathbb{N}\}$ be an orthonormal basis for $\mathcal{C}_2$. Then, $\mathcal{G}$ is a HS-dual frame of $\mathcal{F}$ if and only if the ordinary frame $\{\mathcal{G}_i^*(\nu_{n,m})\}$ be a (ordinary) dual frame of $\{\mathcal{F}_i^*(\nu_{n,m})\}$.*

Proof. Let $\{\mathcal{F}_i\}_{i=1}^{\infty}$ be a HS-frame for $\mathcal{K}$ with respect to $\mathcal{H}$ and $\{\nu_{n,m} : n, m \in \mathbb{N}\}$ be a orthonormal basis of $\mathcal{C}_2$. And let $f_{j,n,m}$ be defined as in (4). The sequence $\{f_{j,n,m}\}$ is a Bessel sequence, since for all $f \in \mathcal{K}$

$$\sum_{n,m\in\mathbb{N}} |\langle f,\ f_{j,n,m}\rangle|^2 = \|\mathcal{F}_j f\|_{\mathcal{HS}}^2 \leq \|\mathcal{F}_j\|^2 \|f\|^2.$$

Now, for any $f \in \mathcal{K}$ and $\mathcal{A} \in \mathcal{C}_2$, we get

$$\begin{aligned}\langle f, \mathcal{F}_j^* \mathcal{A} \rangle &= [\mathcal{F}_i f, \mathcal{A}]_{\mathbf{tr}} \\ &= \left[\sum_{n,m\in\mathbb{N}} \langle f,\ f_{j,n,m} \rangle \nu_{n,m},\ \mathcal{A} \right]_{\mathbf{tr}} \\ &= \left\langle f,\ \sum_{n,m\in\mathbb{N}} [\mathcal{A},\ \nu_{n,m}]_{\mathbf{tr}}\ f_{j,n,m} \right\rangle.\end{aligned}$$

Therefore

$$\mathcal{F}_j^* \mathcal{A} = \sum_{n,m\in\mathbb{N}} [\mathcal{A},\ \nu_{n,m}]_{\mathbf{tr}} f_{j,n,m} \quad (\mathcal{A} \in \mathcal{C}_2). \tag{5}$$

In particular, $\mathcal{F}_j^*(\nu_{n,m}) = f_{j,n,m}$. Therefore by (5), we have

$$\mathcal{F}_j^* \mathcal{A} = \sum_{n,m\in\mathbb{N}} [\mathcal{A},\ \nu_{n,m}]_{\mathbf{tr}}\ \mathcal{F}_j^*(\nu_{n,m}) \quad (\mathcal{A} \in \mathcal{C}_2). \tag{6}$$

Similarly, there exists $g_{j,n,m} \in \mathcal{K}$ such that $\mathcal{G}_j^*(\nu_{n,m}) = g_{j,n,m}$. By Theorem 2.4, $F = \{\mathcal{F}_j^*(\nu_{n,m})\}$ and $G = \{\mathcal{G}_j^*(\nu_{n,m})\}$ are frames for Hilbert space $\mathcal{K}$. Using (6), we obtain that

$$\begin{aligned}T_F T_G^*(f) &= \sum_{i=1}^{\infty} \sum_{n,m\in\mathbb{N}} \langle f, \mathcal{G}_i^*(\nu_{n,m}) \rangle\, \mathcal{F}_i^*(\nu_{n,m}) \\ &= \sum_{i=1}^{\infty} \mathcal{F}_i^* \left(\sum_{n,m\in\mathbb{N}} [\mathcal{G}_i(f), \nu_{n,m}]_{\mathbf{tr}}\ \nu_{n,m} \right) \\ &= \sum_{i=1}^{\infty} \mathcal{F}_i^* \mathcal{G}_i(f) = T_{\mathcal{F}} T_{\mathcal{G}}^*(f).\end{aligned}$$

The proof is completed. □

Suppose that $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for Hilbert space $\mathcal{H}$. As an immediate consequence, we have the following corollary.

Corollary 2.6. *Let $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^{\infty}$ and $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$ be* HS*-frames for $\mathcal{K}$ with respect to $\mathcal{H}$. Then $\mathcal{G}$ is a* HS*-dual frame of $\mathcal{F}$ if and only if $\{\mathcal{G}_i^*(e_n \otimes e_m)\}$ is a (ordinary) dual frame of $\{\mathcal{F}_i^*(e_n \otimes e_m)\}$.*

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