On Module Connes Amenability for $A\times_{\theta} B$ and $A\bigoplus\Omega$ Algebras

Ebrahim tamimi

Department of Mathematics, Velayat University, Iranshahr; e.tamimi@velayat.ac.ir

Abstract
Let $A=(A_*)^*$ and $B=(B_*)^*$ be two Banach algebras where A_* and B_* are preduals of A and B respectively. Let Ω be a normal Banach A-bimodule with predual Ω_* and $\theta : B \to A$ be an algebraic homomorphism. In this paper, we investigate the module Connes amenability for θ-Lau product of Banach algebras. Also, we investigate the conditions for module Connes amenability of θ-Lau product of Banach algebras. Moreover, we characterize the $(\psi,\theta)-$module Connes amenability of θ-Lau product $A\times_{\theta} B$. Finally, we obtain characterization for $(\psi, 0)$-module Connes amenability of module extension of Banach algebra $A\bigoplus\Omega$.

Keywords: module extension Banach algebra, φ-module Connes amenable, module φ-derivation, θ-Lau product.

1. Introduction

Johnson in [11], established the source of term 'amenability' for Banach algebras. The concept of amenability of a Banach algebra A to the case that there exists an additional U-module structure on A, where U is Banac algebra is extended by Amini in [1]. Also, Amini investigate equivalence conditions for module amenability in terms of a special net and an element, called module approximate diagonal and module virtual diagonal respectively. The definition of Connes amenability makes sense for a larger class of Banach algebras. Runde in [19, 20, 21, 22], introduced the concept of Connes amenability of dual Banach algebras and this type of Banach algebras is interested by many researchers. The concept of module Connes amenability for dual Banach algebras which are also Banach modules with respect to the compatible action, first defined by Amini in [2]. It is shown that there exists a relation between module Connes amenability and existance a normal module virtual diagonal. Also, weak amenability of module extension of Banach algebra has further investigated and deepened by Zhang [23].
The θ-Lau product of $A\times_{\theta} B$ where A and B are two Banach algebras and θ is a nonzero multiplicative functional on B was introduced by Lau in [14] for certain class of Banach algebras and followed by Monfared in [16] for the general case.
Let E be a Banach A-bimodule. The collection of all elements of E that module maps from A onto E are w* -weakly continuous, is denoted by $\sigma wc(E)$. A σwc-virtual diagonal for A is an element $\mu \in \sigma wc((A\hat{\bigotimes}A)^*)^*$ such that $a.\mu = \mu.a$ and $a.\Delta_{\sigma wc} = a$ where, $\Delta : A\hat{\bigotimes}A \to A$ is the multiplication operator. In [21], it is shown that a Banach algebra A is Connes amenable if and only if it has a so-called σwc -virtual diagonal. Let A be a Banach algebra, A-bimodule E is called normal if it is a dual space such that the module actions are separately w*-continuous [10, 11]. Recently the authors have introduced the new version, as called φ-Connes amenability of dual Banach algebra A that $\varphi \in \Delta(A)$, the set of all continuous homomorphisms from A onto Complex numbers ¢, and also $\varphi \in A_*$ [8], as follows:
A dual Banach algebra A is φ-Connes amenable if, for every normal A-bimodule E, where the left action is of the form $a.\mathbf{e} = \varphi(a)\mathbf{e}$; $(a \in A,\ \mathbf{e} \in E)$, every bounded w*-continuous derivation $D : A \to E$ is inner.

In this paper, we are going to investigate the module Connes amenability and character module Connes amenability for $A\times_{\theta} B$. Suppose that A is a Banach algebra, F is a subspace of A and $\varphi \in \Delta(A) \cap F$.
A linear functional **m** on F is said to be a mean if

$$\langle \mathbf{m}, \varphi\rangle = 1.$$

A mean **m** is φ-invariant mean if

$$\langle \mathbf{m}, a.f \rangle = \varphi(a)\langle \mathbf{m}, f\rangle$$

for all $a \in A$ and $f \in F$.
Note that φ-Connes amenability of A follows from $F = A_*$ (see [12, 9, 8]).
The authors showed that a dual Banach algebra A is φ-Connes amenable if and only if the second dual, A^{**} has a φ-invariant mean on A_*.

Consider A is a dual Banach algebra, and U is a Banach algebra such that A is a Banach U-bimodule via following actions,

(1) $v.(ab) = (v.\ a).\ b$
(2) $(v.\ w).\ a = v.\ (w.\ a)$
for every $a, b \in A$ and $v, w \in U$.

Let E be a normal dual Banach A-bimodule. Moreover, if E is an U-bimodule via

(1) $v.\ (a.\ \mathbf{e}) = (v.\ a).\ \mathbf{e}$
(2) $(a.\ v).\ \mathbf{e} = a.\ (v.\ \mathbf{e})$
(3) $(v.\ \mathbf{e}).\ a = v.(\mathbf{e}.\ a)$
for every $a \in A$, $v \in U$ and $\mathbf{e} \in E$, then we say that E is a normal Banach left A-U-bimodule. Similarly, for the right

actions. Also, we say that E is symmetric, if v. **e** = **e**. v (v ϵ U, **e** ϵ E).
Let U and A be Banach algebras, and let E be a Banach A-U-bimodule. Then By using [18], E* becomes a Banach U-A-bimodule via

⟨**e**, a. f ⟩ := ⟨**e**. a, f⟩, ⟨**e**, f. v⟩ := ⟨v. **e**, f⟩
For every v ϵ U, a ϵ A, f ϵ E* and **e** ϵ E.

Definition 1.1. Let A be a Banach algebra, U be a Banach algebra such that A is a Banach U-bimodule, φ ϵ Δ(A) ∩ A_* and E be a Banach A-U-bimodule. A bounded map D_U from A to E is called a module φ-derivation if

(1) D_U (v. a ± b. w) = v. D_U (a) ± D_U (b). w;
(2) D_U (ab) = D_U (a). φ(b) + φ(a). D_U (b)
for all a, b ϵ A and v, w ϵ U.

Definition 1.2. Let A be a Banach algebra, U be a Banach algebra such that A is a Banach U-bimodule and φ ϵ Δ(A) ∩ A_*. We say that A is φ-module Connes amenable if for any symmetric normal Banach A-U-bimodule E, each w*-continuous module φ-derivation D_U : A → E is inner.

Module amenability is somehow weaker than amenability. Suppose that A is a Banach U-bimodule and φ is an identity map on Banach algebra A, then id-module Connes amenability of A is equivalent to module Connes amenability of A. Also, by the similar argument of the proof of [1, Proposition 2.1], above notations and that U has a bounded approximate identity for A, it can easily be seen that Connes amenability of A follows its module Connes amenability.

Remark 1.3. In [9], it is defined a certain version of φ-module Connes amenability where φ: A→A is a module homomorphism, as follows:

Let A be a Banach algebra, U be a Banach algebra such that A is a Banach U-bimodule and φ : A → A is a map that satisfied
φ(v. a + b. w) = v. φ(a) + φ(b). w, φ(ab) = φ(a)φ(b),
for every a, b ϵ A; v, w ϵ U.

In this case, A is called φ-module Connes amenable if for any symmetric normal Banach A-U-bimodule E, each w*-continuous module φ-derivation from A to E is inner [9].

2. (ψ , ϴ)-module Connes amenability of ϴ -Lau $A\times_\theta B$

Suppose that A is an unital dual Banach algebra with predual A_*, the identity e_A and B is a dual Banach algebra with predual B_*. Suppose that ψ ϵ Δ(A) ∩ A_*, θ ϵ Δ(B) ∩ B_* and ϴ: B → A is an algebraic homomorphism, which ϴ(b) = θ (b) e_A.
The ϴ-Lau product $A\times_\theta B$ is defined with

$$(a, b).(a, b') = (a. a' + \Theta (b').a + \Theta (b).a, bb')$$

and the norm $\| (a, b) \|_{A\times_\theta B} = \| a \|_A + \| b \|_B$ for all a, a′ ϵ A and b, b′ ϵ B.
This definition is a certain case of the product that is presented in [6, 14, 15, 16]. In this section, we investigate the notions of (ψ , ϴ)-module Connes amenability and (0,ϴ) -module Connes amenability. Since θ ϵ Δ(B) ∩ B_*, then $A\times_\theta B$ is a dual Banach algebra with predual $A_*\times B_*$. It is known that $(A\times_\theta B)^*$ identified with A*× B* that ⟨(f,g), (a, b)⟩ = f(a)+g(b) for all a ϵ A, b ϵ B and f ϵ A*, g ϵ B*.
In this paper we consider that A** and B** are equipped with the first Arens product.

In the following we investigate module Connes amenability of $A\times_\theta B$.

Lemma 2.1. Let A be an unital dual Banach algebra with predual A_* and let B be a dual Banach algebra with predual B_*. Then the following two statements are equevalent,

(i) A and B are id-module Connes amenable.
(ii) $A\times_\theta B$ is id⊗id-module Connes amenable.

Proof. (i) ⇒ (ii) Let U be a Banach algebra, A and B be id-module Connes amenable dual Banach algebras. Moreover, let E be a symmetric normal Banach $A\times_\theta B$-U-bimodule and D_U : $A\times_\theta B$ → E be a bounded w*-continuous module id⊗id -derivation. We show that D_U is inner.
(ii) ⇒ (i) Let $A\times_\theta B$ be id⊗id -module Connes amenable. First, we show that A is id-module Connes amenable. For this purpose, let E be a symmetric normal dual Banach A-U-bimodule and let D_U : A → E be a bounded w*-continuous module id-derivation. By [9, Lemma 3.3], and without loss of generality, suppose that E_* is pseudo-unital. we conclude that D_U is inner.
By the similar argument, B is id-module Connes amenable. ■

Remark 2.2. In [16], it is shown that
Δ($A\times_\theta B$) = {(ψ, θ): ψ ϵ Δ(A)} U { (0, φ) : φ ϵ Δ(B)}
where, θ ϵ Δ(B) ∩ B_*.

Now, in the following theorem we extend Lemma 3. 1.

Theorem 2.3. Let A be an unital dual Banach algebra that is Arens regular and B be a dual Banach algebra. Let ψ ϵ Δ(A) ∩ A_* and θ ϵ Δ(B) ∩ B_*. Then the following statements are hold:

(i) $A\times_\theta B$ is (ψ , ϴ)-module Connes amenable if and only if A is ψ -module Connes amenable.
(ii) $A\times_\theta B$ is (0, ϴ)-module Connes amenable if and only if B is ϴ -module Connes amenable.

Proof. (i) Let A be ψ -module Connes amenable and U be a Banach algebra. Suppose that E is a symmetric normal

Banach A-U-bimodule and $D_U : A \to E$ is a module ψ-derivation. We conclude that A is ψ-module Connes amenable.
Conversely, let E be a symmetric normal Banach A-U-bimodule such that $a.x = \psi(a)x$ for every $a \in A$; $\mathbf{x} \in E$ and let $D_U : A \to E$ be a bounded w*-continuous module ψ-derivation. Suppose that E_* is pseudo-unital. We extend D_U by
$\mathbf{D_U}: A\times_\theta B \to E$, in term

$$(a,b) \to D_U((a,b)(e_A,0)) - (a,b) . D_U(e_A,0)$$

for every $(a, b) \in A\times_\theta B$. As in the proof of Lemma 2.1, $\mathbf{D_U}$ is inner. ■

3. (ψ,0)-module Connes amenability of $A\oplus\Omega$

In this section we study module Connes amenability and character module Connes amenability for module extension of dual Banach algebras. Suppose that A is a dual Banach algebra and $\Omega = (\Omega_*)^*$ is a normal Banach A-bimodule. Indeed, the Banach algebra $A\oplus\Omega$, the l^1-direct sum of a Banach algebra A and a nonzero Banach A-bimodule Ω, with algebraic product

$(a,w) . (a', w') = (aa', aw' + wa')$; $(a, a' \in A$ and $w, w' \in \Omega)$ and defined norm as follows:

$\|(a, w)\| = \| a \|_A + \| w \|_\Omega$;
is called module extension of Banach algebra. Some aspects of algebras of this form have been discussed in [3, 5]. It is known that $A\oplus\Omega$ is a dual Banach algebra with predual $A_* \oplus_\infty \Omega_*$, where $\oplus_\infty$ denotes l_∞-direct sum of Banach A-bimodules.

In [7], the authors showed that $\Delta(A\oplus\Omega) = \Delta(A)\times\{0\}$. They proved that if $A\oplus\Omega$ is (ψ,0)-amenable then A is ψ-amenable and the converse also holds in the case, where $\Omega A = 0$. Here, we extend the result for (ψ, 0)-module Connes amenability. The following theorem is an analog of [13, Theorem 1.1].

Theorem 3.1. Let A be a dual Banach algebra and $\psi \in \Delta(A) \cap A_*$. Let Ω be a normal Banach A-bimodule. Then the following hold:

(i) If $A\oplus\Omega$ is (ψ,0)-module Connes amenable, then A is ψ-module Connes amenable.
(ii) If $\Omega A = 0$ and A is ψ-module Connes amenable, then $A\oplus\Omega$ is (ψ,0)-module Connes amenable.

Proof. (i) Let U be a Banach algebra, and let E be a symmetric normal Banach A-U-bimodule. So, E is a symmetric normal Banach $A\oplus\Omega$-U-bimodule. Let $D_U : A \to E$ be a bounded w*-continuous module ψ-derivation. Consider the projection map $P_U : A\oplus\Omega \to A$. By the assumption the proof is complete.

(ii) Let E be a symmetric normal Banach $A\oplus\Omega$-U-bimodule. Since A is a normal Banach U-bimodule, then E is a symmetric normal Banach A-U-bimodule. Suppose that $\mathbf{D_U}: A \to E$ be a bounded w*-continuous module ψ-derivation and $D_U: A\oplus\Omega \to E$ be a bounded w*-continuous module (ψ, 0)-derivation. We show that D_U is inner. ■

Corollary 3.2. Let A be a dual Banach algebra with predual A_* and let Ω be a normal Banach A-bimodule with pseudo-unital predual Ω_*. Then the following are equivalence:

(i) $A\oplus\Omega$ is id-module Connes amenable.
(ii) A is id-module Connes amenable.

Proof. The statements follow from Lemma 2.1 and Theorem 3.1 and by using [4,17]. ■

Conclusions

In this paper, we investigate the conditions for module Connes amenability of Θ-Lau product of Banach algebras. Moreover, we characterize the (ψ, Θ)-module Connes amenability of Θ-Lau product $A\times_\theta B$. Finally, we obtain characterization for (ψ, 0)-module Connes amenability of module extension of Banach algebra $A\oplus\Omega$.

Acknowledgement

The author thanks the referees very much for their precious time and valuable comments.

References

[1] Amini, M., 2004. "Module amenability for semigroup algebras". *Semigroup Forum*, pp. 243-254.
[2] Amini, M., 2015. "Module Connes amenability of hypergroup measure algebras". *Open Math*, 13, pp. 737-756.
[3] Bade, W. G., Dales, H. G., and Lykova, Z. A., 1999. "Algebraic and strong splittings of extensions of Banach algebras". *Mem. Amer. Math. Soc*, 137(656), 113pp.
[4] Bonsall, F. F., and Duncan, J., 1973. Complete Normed Algebras, Springer-Verlag, Berlin.
[5] Dales, H. G., Ghahramani, F., and Gronbaek, N., 1998. "Derivations intoiterated duals of Banach algebras". *Studia Math*, MR 99g:46064, 128, pp. 19-54.
[6] Daws, M., 2006. "Connes amenability of bidual and weighted semigroup algebras". *Math. Scand*, 99, pp. 217-246.
[7] Ebrahimi Vishki, H. R., and Khoddami, A. R., 2011. "Character inner amenability of certain Banach algebras". *Colloquium Mathematicum*, 122, pp. 225-232.
[8] Ghaffari, A., and Javadi, S., 2017. "φ-Connes amenability of dual Banach algebras". *Bull. Iranian Math. Soc*, 43, pp. 25-39.
[9] Ghaffari, A., and Javadi, S., and Tamimi, E., 2020. "φ-Connes module amenability of dual Banach algebras". *Journal of Algebraic Systems*, 8(1), pp. 69-82.

[10] Johnson, B. E., 1972. "Cohomology in Banach algebras". *Mem. Amer. Math. Soc*, 127.

[11] Johnson, B. E., Kadison, R. V., and Ringrose, J., 1972. " Cohomology of operator algebras III". *Bull. Soc. Math. France*, 100, pp. 73-79.

[12] Kaniuth, E. A., Lau, T., and Pym, J., 2008. "On φ-amenability of Banach algebras". *Math. Proc. Cambridge Philos. Soc*, 144(1), pp. 85-96.

[13] Kaniuth, E. A., Lau, T., and Pym, J., 2008. "On character amenability of Banach algebras". *J. Math. Anal. Appl*, 344, pp. 942-955.

[14] Lau, A. T., 1983. "Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups". *Fund. Math,* 118, pp. 161-175.

[15] Monfared, M. S., 2008. "Character amenability of Banach algebras". *Math. Proc. Camb. Phil. Soc*, 144, pp. 697-706.

[16] Monfared, M. S., 2007. "On certain products of Banach algebras with applications to harmonic analysis". *Studia Math*, 178, pp. 277-294.

[17] Rudin, W., 1991. *Functional analysis*, McGraw Hill, New York.

[18] Runde, V., 2001. "Amenability for dual Banach algebras". *Studia Math*, 148(1), pp. 47-66.

[19] Runde, V., 2003. "Connes amenability and normal virtual diagonals for measure algebras I". *J. Lond. Math. Soc*, 67(3), pp. 643-656.

[20] Runde, V., 2003. "Connes amenability and normal virtual diagonals for measure algebras II". *Bull. Aust. Math. Soc*, 68(2), pp. 325-328.

[21] Runde, V., 2004. "Dual Banach algebras: Connes amenability, normal, virtual diagonals, and injectivity of the predual bimodule". *Math. Scand*, 95(1), pp. 124-144.

[22] Runde, V., 2002. Lectures on amenability, Lecture Notes in Mathematics. Springer- Verlag, Berlin.

[23] Zhang, Y., 2002. "Weak amenability of module extensions of Banach algebras". *Trans. Amer. Math. Soc.* 354, pp. 4131-4151.