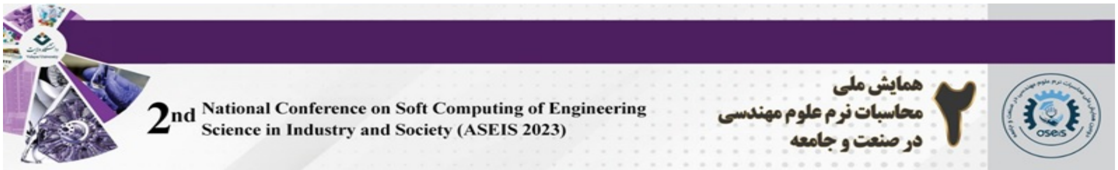

A New Distorted Weibull Distribution with Application to Reliability

M. Esfahani[1,*]

[1]Department of Statistics, Velayat University of Iranshahr, Iranshahr, Iran;
M.Esfahani@velayat.ac.ir

ABSTRACT. Distortion functions are popular models for generating new distributions. In this article, a new family of distributions is introduced by using the distortion function method. Then the features of this introduced family are stated and it is also studied from the point of view of reliability and weighted functions. In the following, considering the basic Weibull distribution, a new generalization of the Weibull distribution is presented and it is called the new distortion Weibull distribution. In the end, by analyzing a real data, it is shown that the newly introduced distribution has a better fit to the lifetime data than the basic distribution.

Keywords: Weibull Distribution, Distortion Function, Failure Rate Function, Reliability Theory and Weighted Function.

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1. Introduction

One of the most generally used analyzed life time, reliability and economics is Weibull distribution. It has flexible failure rate function, and it can be increasing, decreasing and constant for different parameters. Considering the importance of the failure rate function behavior in lifetime data modeling, it is interesting to find distributions with high flexibility in the failure rate function. So that its failure rate function is unimodal and bathub-shape in addition to the mentioned states.

In the following, some of the definitions required in the article are briefly stated. In section 2, a new family of distribution functions is introduced and its properties are stated. In section 3, considering the Weibull distribution as the base distribution, a new distribution called the new distorted Weibull distribution (NDW) is introduced and its reliability features are stated. Finally, it is shown by analyzing the lifetime data that the new distribution provides a better fit than the basic Weibull distribution.

DEFINITION 1.1. Random variable X has Weibull distribution if it has probability density function and distribution function in the following form, respectively;

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*Speaker

$$f(x) = \alpha\beta^{-\alpha}x^{\alpha-1}\exp\{-(\frac{x}{\beta})^{\alpha}\},$$

$$F(x) = 1 - \exp\{-(\frac{x}{\beta})^{\alpha}\}, x < 0, \quad \alpha, \beta > 0. \tag{1}$$

[1] and [2] are complete references on the Weibull distribution.

Definition 1.2. The function $g(x) : [0, 1] \longrightarrow [0, 1]$ is a distortion function whenever $g(0) = 0$, $g(1) = 1$ and $g(.)$ be a non-decreasing continuous function on $[0, 1]$.

For more details about the distortion function and its features, see [3], [4] and [5].

2. Proposed model

In this section, we first introduce a family of distribution functions by using the distortion function method and then describe some of its reliability features. In the following, we study the produced family from the point of view of weighted distributions as well as reliability theory.

[4] showed that if random variable X has distribution function $F_X(x)$ and $g(.)$ be a distortion function then the transformation of distribution function F by means of $g(.)$ is

$$F_g(x) = g[F_X(x)] = goF_X(x), \tag{2}$$

where $F_g(x)$ is also a distribution function and is called distorted distribution F. In the following, we will make a distorted distribution using this method. Let N be a discrete random variable with the following probability function;

$$P(N = k) = \frac{\theta^{k-1}e^{-\theta}}{(k-1)!}, \quad \theta \geq 0, \quad k = 1, 2, 3, \ldots. \tag{3}$$

If we obtain the probability-generating function of random variable N, by performing a simple calculation, for $0 \leq t \leq 1$ we have:

$$g_N(t) = E(t^N) = t\exp(-\theta(1-t)), \quad \theta > 0, \tag{4}$$

where in $g_N(t)$ is a distortion function since $g_N(t) : [0, 1] \to [0, 1]$ is a increasing function and $g_N(0) = 0$ and $g_N(1) = 1$. Now considering $t = F(x)$, we have

$$F_g(x) = g_N(F(x)) = F(x)\exp(-\theta\bar{F}(x)), \tag{5}$$

is a distribution function that and is called the distorted distribution function F. It is clear that if $\theta \longrightarrow 0$ then

$$F_g(x) \longrightarrow F(x).$$

Remark 2.1. Let $F_g(x)$ be a distribution of the form (5). Then failure rate function of $F_g(x)$ is

$$r_g(x) = \frac{f_g(x)}{\bar{F}_g(x)} = \frac{f(x)(\theta + e^{-\theta\bar{F}(x)})}{1 - F(x)\exp(-\theta\bar{F}(x))}.$$

2.1. A New distorted weibull distribution. In following, by using model (5) and placing the Weibull distribution introduced in (1), we make a new distribution, which is called the new distorted Weibull distribution and is denoted by NDW.

According to the above assumptions, we say that X has NDW distribution if its density function and distribution function are respectively as follows;

$$f_{NDW}(x) = \alpha\beta^{-\alpha}x^{\alpha-1}\exp\{-(\frac{x}{\beta})^{\alpha}\}\exp(-\theta\exp\{-(\frac{x}{\beta})^{\alpha}\})(1+\theta(1-\exp\{-(\frac{x}{\beta})^{\alpha}\})),$$

$$F_{NDW}(x) = (1-\exp\{-(\frac{x}{\beta})^{\alpha}\})\exp(-\theta\exp\{-(\frac{x}{\beta})^{\alpha}\}), x>0, \quad \alpha,\beta,\theta>0, \tag{6}$$

and we write $X \sim NDW(\alpha,\beta,\theta)$.

The failure rate function of distribution (6) is

$$r_g(x) = \frac{\alpha\beta^{-\alpha}x^{\alpha-1}\exp\{-(\frac{x}{\beta})^{\alpha}\}\exp(-\theta\exp\{-(\frac{x}{\beta})^{\alpha}\})(1+\theta(1-\exp\{-(\frac{x}{\beta})^{\alpha}\}))}{1-(1-\exp\{-(\frac{x}{\beta})^{\alpha}\})\exp(-\theta\exp\{-(\frac{x}{\beta})^{\alpha}\})}.$$

By plotting this function for different parameter values, its behavior can be checked.

2.2. Parameters estimation of NDW model. The method of obtaining the estimation of the parameters of model (6) using the maximum likelihood method is explained.

Suppose $X_1, X_2, ..., X_n$ are random variables with observed values $x_1, x_2, ..., x_n$ from distribution (6). In this case, the likelihood function for this distribution is as follows:

$$L(x_1,x_2,...x_n,\alpha,\beta,\theta) = \Pi_{i=1}^{n}\alpha\beta^{-\alpha}x_i^{\alpha-1}\exp\{-(\frac{x_i}{\beta})^{\alpha}\}\exp(-\theta\exp\{-(\frac{x_i}{\beta})^{\alpha}\})$$
$$\times(1+\theta(1-\exp\{-(\frac{x_i}{\beta})^{\alpha}\})),$$

and as a result, the logarithm of the likelihood function will be as follows:

$$\begin{aligned} l(x_1,x_2,...x_n,\alpha,\beta,\theta) &= \ln L(x_1,x_2,...x_n,\alpha,\beta,\theta) \\ &= \sum_{i=1}^{n}\ln\alpha - \alpha\ln\beta + (\alpha-1)\ln x_i \\ &-(\frac{x_i}{\beta})^{\alpha} - \theta\exp\{-(\frac{x_i}{\beta})^{\alpha}\} + \ln(1+\theta(1-\exp\{-(\frac{x_i}{\beta})^{\alpha}\})). \end{aligned}$$

By deriving the logarithm of the likelihood function with respect to each of parameters α, β, and θ and setting them equal to zero, the maximum likelihood estimator can be obtained for each of the parameters.

$$\begin{aligned} \frac{\partial l}{\partial\alpha} &= \sum_{i=1}^{n}\frac{1}{\alpha} - \ln\beta + \ln x_i - (\frac{x_i}{\beta})^{\alpha}\ln(\frac{x_i}{\beta}) \\ &+\theta(\frac{x_i}{\beta})^{\alpha}\ln(\frac{x_i}{\beta})\exp\{-(\frac{x_i}{\beta})^{\alpha}\} + \frac{\theta(\frac{x_i}{\beta})^{\alpha}\ln(\frac{x_i}{\beta})\exp\{-(\frac{x_i}{\beta})^{\alpha}\}}{1+\theta(1-\exp\{-(\frac{x_i}{\beta})^{\alpha}\})} = 0, \\ \frac{\partial l}{\partial\beta} &= \sum_{i=1}^{n} -\frac{\alpha}{\beta} + \frac{(\alpha+1)x_i^{\alpha}}{\beta^{\alpha+1}} \end{aligned}$$

$$-\theta\exp\{-(\frac{x_i}{\beta})^{\alpha}\}\frac{(\alpha+1)x_i^{\alpha}}{\beta^{\alpha+1}} - \frac{\theta\exp\{-(\frac{x_i}{\beta})^{\alpha}\}\frac{(\alpha+1)x_i^{\alpha}}{\beta^{\alpha+1}}}{1+\theta(1-\exp\{-(\frac{x_i}{\beta})^{\alpha}\})} = 0,$$

$$\frac{\partial l}{\partial\theta} = \sum_{i=1}^{n} -\exp\{-(\frac{x_i}{\beta})^{\alpha}\} + \frac{1-\exp\{-(\frac{x_i}{\beta})^{\alpha}\}}{1+\theta(1-\exp\{-(\frac{x_i}{\beta})^{\alpha}\})} = 0.$$

But as see, the obtained equations do not have a closed form and are not easily solved, and it is possible to solve these equations using mathematical and statistical software.

3. Application

In this section two applications of model (5) are stated and the presented model is examined from the point of view of weighted distributions as well as reliability theory.

3.1. Weighted distributions viewpoint. The distribution family obtained in (5) can also be interpreted from the point of view of weighted distributions. For this purpose let $F_g(x)$ be a disttorted distribution function (5). Then it is simply shown that $f_g(x) = (1+\theta\exp(-\theta\bar{F}(x)))f(x)$ is the density of distorted distribution function. Noe with considering $f_g(x) = w(x)f(x)$ where in $w(x) = 1+\theta\exp(-\theta\bar{F}(x))$ is a increasing weighted function. Therefore if X_w has a distribution of $F_g(x)$, then X_w is a weighted random variable with density function $f_g(x)$.

3.2. Reliability viewpoint. To study distorted distribution (5) from the perspective of reliability, let $\{X_n; n = 1, 2, ...\}$ a sequence of non-negative, independent and identically distributed random variables with distribution function F and N is a discrete random variable with probability function (3) and independent of X_i. If the new random variable Z is considered as $Z = \max\{X_1, ..., X_N\}$ then

$$\begin{aligned} G(z) = P(Z \le z) &= \sum_{n=1}^{\infty} P(Z \le z|N = n)P(N = n) = \sum_{n=1}^{\infty} F^n(z)P(N = n) \\ &= \sum_{n=1}^{\infty} F^n(z)\frac{\theta^{n-1}e^{-\theta}}{(n-1)!} = e^{-\theta}F(z)\sum_{n=1}^{\infty}\frac{(\theta F(z))^{n-1}}{(n-1)!} \\ &= e^{-\theta}F(z)\exp\{\theta F(z)\} = F(z)\exp\{-\theta F\bar{(}z)\}, \end{aligned}$$

which is the same as $F_g(x)$ obtained in (5). Therefore, if $X_1, ..., X_N$ are the lifetimes of the components of a parallel system with N components, where N is a discrete random value with a probability function (3) and X_i are independent and identically distribution random variables with distribution F, which are also independent of N, then $Z = \max\{X_1, ..., X_N\}$ is the lifetime of this parallel system which has a distribution function (5).

4. Real data analyze

In this section, we use a real data analysis to show the validity of the NDW model. The study data set is 50 samples of Breaking Stresses of Carbon Fibers data employ in [6]. Corresponding fitted criteria are given by NDW and Weibull distributions. Fit test criteria shown in Tables 1. It is clear that the NDW distribution is a suitable

Table 1. The goodness-of-fit measures for Breaking Stresses of Carbon Fibers data.

Model	AIC	CAIC	BIC	HQIC	KS	P-value
NDW	100.51	101.03	106.24	102.61	0.0902	0.8103
$Weibull$	104.15	104.41	107.97	105.61	0.1299	0.3676

Table 2. MLE values of the parameters.

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$
$Weibull$	3.21	4.78	$---$
NDW	2.33	2.01	6.51

distribution for these data and also better from Weibull distribution to fit these data set. Table 2 show the ML estimator of unknown parameters of two distributions.

5. Conclusion

In this article, using the distortion function technique, a new family of distributions has been introduced and its features and applications have been investigated. Then, considering the Weibull distribution as the basic distribution, a new distribution is presented and some of its characteristics are stated. In the end, the superiority of the newly introduced distribution over the basic Weibull distribution is shown by using a real data set.

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