2nd National Conference on Soft Computing of Engineering Science in Industry and Society (ASEIS 2023)

۲ همایش ملی
محاسبات نرم علوم مهندسی
در صنعت و جامعه

Solving the Inverse Kinematic Robotics Problem Using Parametric Gröbner Bases

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Abstract. This paper utilizes computer algebra techniques to solve the inverse kinematics problem of two simple planar robots and the 3D Romin robot.

Keywords: Inverse kinematics, Romin robot, Monomial ordering, Gröbner systems.

AMS Mathematics Subject Classification [2010]: 13P10, 13P25

1. Introduction

The inverse kinematics problem (IKP) uses kinematic equations to determine the motion of a robot to reach a desired position. Since these equations depend on some parameters and variables, it is useful to utilize the parametric Gröbner bases for solving these parametric equations to find the possible ways to reach a fixed position. The concept of Gröbner bases is a particular kind of generating set of a polynomial ideal and is a powerful tool in computer algebra. Gröbner bases were introduced in 1965, together with an algorithm (Buchberger's algorithm) to compute them by Bruno Buchberger in his Ph.D. thesis [2]. He named them after his advisor Wolfgang Gröbner. Then, he proposed [1] two criteria to enhance the performance of his algorithm. In 1983, Lazard described an algorithm for computing Gröbner bases, by using linear algebra techniques [13]. Later on, Faugère propounded his two famous algorithms namely F_4 and F_5 for computing Gröbner bases (see [4,5]). Since our study focuses on parametric Gröbner bases, namely the Gröbner system, briefly, we review the existing literature on this topic. The concept of *Gröbner systems* (and also the first algorithm to compute them) was introduced by Weispfenning in [16]. Finally, Kapur et al. in 2010 proposed most efficient algorithm (PGBMain algorithm) for computing them [12]. In these years, effective studies and many results were obtained in the computations of Gröbner systems [3,7–9,11,15]. Gröbner systems have numerous applications in Mathematics and other field of sciences. In particular, we can point out algebraic geometry [7,14–16], parametric linear algebra [3,10], robotics [14], automated geometry theorem proving [15], and so on. In continue, we review the basic definitions and notations of Gröbner systems. Throughout this text, we consider $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring in terms of $x_1, \ldots, x_n$ over a field $\mathbb{K}$. Let $\mathcal{I} = \langle f_1, \ldots, f_k \rangle \subset \mathcal{R}$ be the

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polynomial ideal generated by the f_i's. We consider a monomial ordering $\prec$ on the set of all monomials (power products of the x_i's) of $\mathcal{R}$. For any $f \in R$, the *leading monomial* of f, denoted by $\mathrm{LM}_\prec(f)$, is the greatest monomial (with respect to $\prec$) appearing in f and its coefficient is the *leading coefficient* of f which denoted by $\mathrm{LC}_\prec(f)$. The *leading term* of f with respect to $\prec$ is the product $\mathrm{LT}_\prec(f) = \mathrm{LC}_\prec(f)\mathrm{LM}_\prec(f)$. The *leading monomial ideal* of $\mathcal{I}$ is defined to be $\mathrm{LM}_\prec(\mathcal{I}) = \langle \mathrm{LM}_\prec(f) \mid f \in \mathcal{I}\rangle$. A finite subset $\{g_1, \dots, g_m\} \subset \mathcal{I}$ is called a *Gröbner basis* for $\mathcal{I}$ with respect to $\prec$ if $\mathrm{LM}_\prec(\mathcal{I}) = \langle \mathrm{LM}_\prec(g_1), \dots, \mathrm{LM}_\prec(g_m)\rangle$.

Using these notations, we recall the definition of Gröbner bases for parametric polynomial ideals, namely Gröbner systems. Roughly speaking, Gröbner systems can be considered as an extension of Gröbner bases for polynomial ideals over fields to polynomial ideals with parametric coefficients. With more details, let us consider $\mathcal{S} = \mathbb{K}[\mathbf{a}, \mathbf{x}]$ as a polynomial ring with parametric coefficients where $\mathbf{a} = a_1, \dots, a_m$ is a sequence of parameters, $\mathbf{x} = x_1, \dots, x_n$ is a sequence of variables and $\{\mathbf{x}\} \cap \{\mathbf{a}\} = \emptyset$. Thus a monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is denoted by x^α where $\alpha = (\alpha_1, \dots, \alpha_n)$. Let $\prec_\mathbf{x}$ be a monomial ordering on the variables and $\prec_\mathbf{a}$ a monomial ordering on the parameters. For defining Gröbner systems, we shall need also to give recall a product ordering to specify an ordering on $\mathcal{S}$. The product of $\prec_\mathbf{x}$ and $\prec_\mathbf{a}$ denoted by $\prec_{\mathbf{x},\mathbf{a}}$, is defined as follows: For all $\alpha, \beta \in \mathbb{N}^n$ and $\gamma, \delta \in \mathbb{N}^m$, we write $\mathbf{x}^\alpha \mathbf{a}^\gamma \prec_{\mathbf{x},\mathbf{a}} \mathbf{x}^\beta \mathbf{a}^\delta$ if either $\mathbf{x}^\alpha \prec_\mathbf{x} \mathbf{x}^\beta$ or ($\mathbf{x}^\alpha = \mathbf{x}^\beta$and $\mathbf{a}^\gamma \prec_\mathbf{a} \mathbf{a}^\delta$). In addition, if $\overline{\mathbb{K}}$ denotes the algebraic closure of $\mathbb{K}$ then from a specialization of parameters we mean a morphism $\sigma : \mathbb{K}[\mathbf{a}] \to \overline{\mathbb{K}}$. Therefore, for each f, we can write $\sigma(f) = f|_{\mathbf{a}=t_1,\dots,t_m}$ where $\sigma(a_i) = t_i$. Furthermore, we say that a specialization σ satisfies $(N, W) \subset \mathbb{K}[\mathbf{a}] \times \mathbb{K}[\mathbf{a}]$ if $\sigma(p) = 0$ for all $p \in N$ and $\sigma(q) \neq 0$ for some $q \in W$ (N_i and W_i are called the null and non-null condition sets, respectively.). Equivalently, σ satisfies (N, W) if $(t_1, \dots, t_m) \in \mathbb{V}(N) \setminus \mathbb{V}(W)$ where $\sigma(a_i) = t_i$. If $\mathbb{V}(N) \setminus \mathbb{V}(W) = \emptyset$ then (N, W) is said to be *inconsistent*. Also, the set of common zeros of $N \subset \mathcal{R}$ is denoted by $\mathbb{V}(N)$, for a set of polynomials N.

Definition 1.1. Let $F \subset \mathcal{S}, G_i \subset \mathcal{S}$ and $(N_i, W_i) \subset \mathbb{K}[\mathbf{a}] \times \mathbb{K}[\mathbf{a}]$ for $i = 1, \dots, \ell$. The triple set $\mathscr{G} = \{(N_i, W_i, G_i)\}_{i=1}^{\ell}$ is called a Gröbner system for $\langle F\rangle$ with respect to $\prec_{\mathbf{x},\mathbf{a}}$ over $\mathcal{V} \subseteq \overline{\mathbb{K}}^m$ if for any i we have

- $\sigma(G_i) \subset \overline{\mathbb{K}}[\mathbf{x}]$ is a Gröbner basis of $\langle\sigma(F)\rangle$ with respect to $\prec_\mathbf{x}$, for any specialization $\sigma : \mathbb{K}[\mathbf{a}] \to \overline{\mathbb{K}}$ satisfying (N_i, W_i)
- $\mathcal{V} \subseteq \bigcup_{i=1}^{\ell} \mathbb{V}(N_i) \setminus \mathbb{V}(W_i)$.

For each i, (N_i, W_i, G_i) is called a branch (segment) of the Gröbner system $\mathscr{G}$. Furthermore, if $\mathcal{V} = \overline{\mathbb{K}}^m$ then $\mathscr{G}$ is called a Gröbner system of F.

Example 1.2. Let $F = \{(a-1)x+bxy+1, y+(b-1)x+a\} \subset \mathbb{K}[a, b, x, y]$ where x, y are variables and a, b are parameters. Using our implementation of the PGBMain algorithm, we can compute a Gröbner system of $\langle F\rangle$ w.r.t. $y \prec_{drl} x$ and $b \prec_{drl} a$ as follows

$$\begin{cases} ([\ ], & [b(b-1)], & [aby + by^2 + a^2 + ay - a - b - y + 1, bx + a - x + y]) \\ ([b], & [a-1], & [a^2 + ay - a - y + 1, x - y - a]) \\ ([b-1], & [\ ], & [y + a, x - 1])) \\ (\text{Other cases}, & & [1])). \end{cases}$$

For instance, if we set $a = 1$ and $b = 2$ then the first branch corresponds to these values of parameters and so $\{2y^2 + 2y - 1, x + y + 1\}$ is a Gröbner basis for the ideal $\langle F\rangle\ |_{a=1,b=2}$.

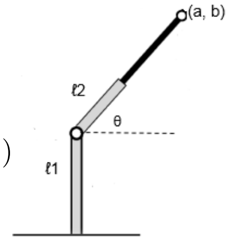
Solving the IKP of two simple planar robots.

The first robot we will consider in this part to solve IKP is composed of one revolute joint and two arms. The arm with length l_1 is fixed at the origin and another has length l_2.

Applying the rotation matrix and also using the trigonometric formula $\sin^2 x + \cos^2 x = 1$ and also, via the substitutions $s = \sin\theta$ and $c = \cos\theta$ arises the following system of equations.

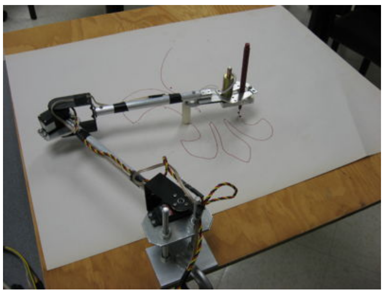
$$a = (l_1 + l_2)c, \quad b = (l_1 + l_2)s, \quad s^2 + c^2 - 1 = 0.$$

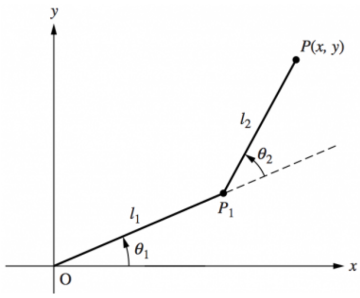
The all possible ways to reach a fixed position $(a, b) \in R^2$ are represented by solving these parametric equations using the concept of Gröbner system. For this purpose, we compute a Gröbner system of $F = \{a - (l_1 + l_2)c, b - (l_1 + l_2)s, s^2 + c^2 - 1\} \subset \mathbb{C}[l_1, l_2, a, b][c, s]$ w.r.t. lexicographical ordering on variables and parameters.



$$\left\{ \begin{array}{lll} ([(l_1 + l_2)^2 - (a^2 + b^2)], & [a, l_1 + l_2], & [(l_1 + l_2)s - b, ac + bs - l_1 - l_2]) \\ ([(l_1 + l_2)^2 - b^2, a], & [b], & [bs - l_1 - l_2, bc]) \\ ([l_1 + l_2, a, b], & [\,], & [c^2 + s^2 - 1])) \end{array} \right.$$

For example, the first segment of the above system shows that if $(l_1+l_2)^2-(a^2+b^2) = 0$ and $a \neq 0, l_1+l_2 \neq 0$ then $l_1s+l_2s-b = 0$ and $ac+bs-l_1-l_2 = 0$ which deduce that $s = \frac{b}{l_1+l_2}$ and $c = \frac{(l_1+l_2)^2-b^2}{a(l_1+l_2)}$. Therefore, $\theta = \sin^{-1}(\frac{b}{l_1+l_2})$ or equivalently, $\theta = \cos^{-1}(\frac{(l_1+l_2)^2-b^2}{a(l_1+l_2)})$. For instance, if we set $l_1 = 2, l_2 = 3$ and $(a, b) = (3, 4) \in R^2$ then the first branch corresponds to these values of parameters and so $\{-4+5s, 3c+4s-5\}$ is a Gröbner basis for the ideal $\langle F \rangle \mid_{l_1=2,l_2=3,a=3,b=4}$. Thus, $s = \sin\theta = \frac{4}{5}$ and so $\theta = \sin^{-1}(\frac{4}{5}) = \cos^{-1}(\frac{3}{5}) = 53.13010233°$.





Now, consider a planar robot that is formed from two revolute joints and two arms. Gröbner system shows how to obtain the kinematics of a 2-link robotic arm in order to solve for the joint angles given an arbitrary end-effector position. The inverse kinematics problem is supplied by the solution of the following system of equations:

$$l_1c_1 + l_2(c_1c_2 - s_1s_2) = x, \quad l_1s_1 + l_2(s_1c_2 + c_1s_2) = y, \quad s_1^2 + c_1^2 = 1, \quad s_2^2 + c_2^2 = 1$$

This system is obtained using the coordinate transformation matrix and the trigonometric formula $\sin^2 x + \cos^2 x = 1$, via the substitutions $s_i = \sin\theta_i$ and $c_i = \cos\theta_i$ for $i = 1, 2$. Consequently, $F = \{l_1c_1+l_2(c_1c_2-s_1s_2)-x, l_1s_1+l_2(s_1c_2+c_1s_2)-y, s_1^2+c_1^2-1, s_2^2+c_2^2-1\}$ is raised as a subset of $\mathbb{C}[l_1, l_2, x, y][c_1, s_1, c_2, s_2]$. So, a Gröbner system of F w.r.t. $x \prec_{lex} y \prec_{lex} l_2 \prec_{lex} l_1$ and $s_2 \prec_{lex} c_2 \prec_{lex} s_1 \prec_{lex} c_1$ is computed as follows (It is worth noting that a Gröbner system of these generators is computed in [14] while the length of fixed l_1 is assumed to be one. Against, here l_1 is the length of a prismatic joint that is considered a variant parameter)

Parametric Constraints	Gröbner Bases
$[\ ]$, $[-4xl_1^2l_2^2(x^2+y^2)(l_1^2+l_2^2-x^2-y^2)]$	$[4l_1^2l_2^2s_2^2+l_1^4-2l_1^2l_2^2-2l_1^2x^2-2l_1^2y^2+l_2^4-2l_2^2x^2-2l_2^2y^2+x^4+2x^2y^2+y^4,$ $2l_1l_2s_2^2-c_2l_1^2-c_2l_2^2+c_2x^2+c_2y^2-2l_1l_2, -c_2l_2y+l_2s_2x+s_1x^2+s_1y^2-l_1y,$ $c_1x-c_2l_2+s_1y-l_1]$
$[x], [l_1, l_2, y, -l_1^2-l_2^2+y^2]$	$[4l_1^2l_2^2s_2^2+l_1^4-2l_1^2l_2^2-2l_1^2y^2+l_2^4-2l_2^2y^2+y^4,$ $2l_1l_2s_2^2-c_2l_1^2-c_2l_2^2+c_2y^2-2l_1l_2, -c_2l_2+s_1y-l_1, c_1y-l_2s_2]$
$[-l_1^2+l_2^2, y, x], [l_2]$	$[l_2s_2, c_2l_2+l_1, c_1^2+s_1^2-1]$
$[l_1, l_2, y, x], [1]$	$[c_2^2+s_2^2-1, c_1^2+s_1^2-1]$
$[l_1^2l_2^2, -l_1^3l_2-l_1l_2^3+l_1l_2y^2,$ $l_1^4-2l_1^2y^2+l_2^4-2l_2^2y^2+y^4, x],$ $[-l_1^4+l_1^2y^2-l_2^4+l_2^2y^2]$	$[2l_1l_2s_2^2-c_2l_1^2-c_2l_2^2+c_2y^2-2l_1l_2, -c_2l_2+s_1y-l_1, c_1y-l_2s_2]$
$[l_1l_2, -l_1^2-l_2^2+y^2, x], [l_1^4+l_2^4]$	$[c_2^2+s_2^2-1, -c_2l_2+s_1y-l_1, c_1y-l_2s_2]$
$[-l_1^2-l_2^2+y^2, x], [l_1, l_2, l_1^2+l_2^2]$	$[l_1l_2s_2^2-l_1l_2, l_1l_2c_2, -c_2l_2+s_1y-l_1, c_1y-l_2s_2]$
$[l_1l_2, -l_1^2-l_2^2+x^2+y^2],$ $[l_2, -l_2+y, l_2+y]$	$[c_2^2+s_2^2-1, -c_2l_2y+l_2^2s_1+l_2s_2x, c_1x-c_2l_2+s_1y-l_1]$
$[l_1l_2, l_2^2, l_2y, l_2x, -l_1^2+x^2+y^2],$ $[l_1, x, -l_1+y, l_1+y]$	$[c_2^2+s_2^2-1, l_1s_1x-xy, c_1x-c_2l_2+s_1y-l_1]$
$[x^2+y^2], [l_1, l_2, y, -l_1+l_2, l_1+l_2]$	$[2l_1l_2s_2x-l_1^2y+l_2^2y, c_2l_2x+l_2s_2y+l_1x,$ $-l_1^2l_2s_2-2l_1^2s_1x+l_2^3s_2+2l_2^2s_1x+2l_1xy, c_1x-c_2l_2+s_1y-l_1]$
$[-l_1^2-l_2^2+x^2+y^2],$ $[l_1, l_2, l_1^2+l_2^2, -l_1^2-l_2^2+y^2]$	$[l_1l_2s_2^2-l_1l_2, l_1l_2c_2, -c_2l_2y+l_1^2s_1+l_2^2s_1+l_2s_2x-l_1y, c_1x-c_2l_2+s_1y-l_1]$

According to this system, we could choose all possible rotation angles for the revolute joints and different lengths for the prismatic joints to achieve some position (x, y) in the plane. For instance, the last branch expresses that if $-l_1^2-l_2^2+x^2+y^2=0$ and $l_1\neq 0, l_2\neq 0, l_1^2+l_2^2\neq 0, -l_1^2-l_2^2+y^2\neq 0$ then the following system of equations arises:

$$\begin{cases} l_1l_2s_2^2-l_1l_2 &= 0\\ l_1l_2c_2 &= 0\\ -c_2l_2y+l_1^2s_1+l_2^2s_1+l_2s_2x-l_1y &= 0\\ c_1x-c_2l_2+s_1y-l_1 &= 0 \end{cases}$$

which deduce the following two solution sets

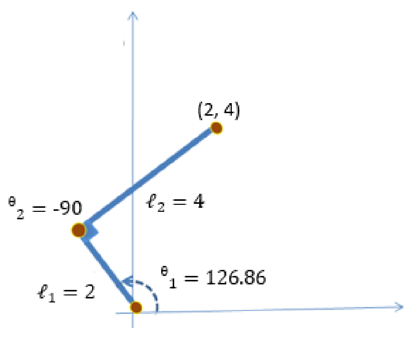
$$\{\quad s_2=1,\qquad c_2=0,\quad s_1=\frac{(l_1y-l_2x)}{(l_1^2+l_2^2)},\quad c_1=\frac{(l_1^3+l_1l_2^2-l_1y^2+l_2xy)}{(l_1^2+l_2^2)x}\}$$
$$\{\quad s_2=-1,\quad c_2=0,\quad s_1=\frac{(l_1y+l_2x)}{(l_1^2+l_2^2)},\quad c_1=\frac{(l_1^3+l_1l_2^2-l_1y^2-l_2xy)}{(l_1^2+l_2^2)x}\}$$

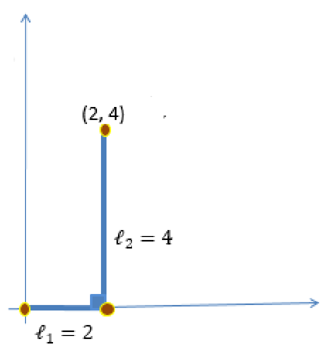
If $l_1=2, l_2=4$ be the lengths of two arms, and if $(x,y)=(2,4)$ be the arm's end-effector position then these values satisfy the parametric conditions at the last branch. Thus,

$$\{s_2=1,\qquad c_2=0,\quad s_1=0,\quad c_1=1\qquad\}$$
$$\{s_2=-1,\quad c_2=0,\quad s_1=\frac{4}{5},\quad c_1=\frac{-3}{5}\quad\}$$

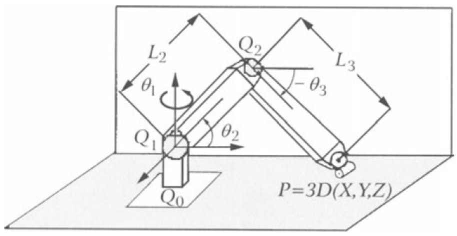
Therefore, the possible rotation angles equal to

$$\begin{cases} \theta_1=\sin^{-1}(0)=\cos^{-1}(1)=0^\circ & \textbf{and} \quad \theta_2=\sin^{-1}(1)=\cos^{-1}(0)=90^\circ\\ & \textbf{or}\\ \theta_1=\sin^{-1}(\frac{4}{5})=\cos^{-1}(-\frac{3}{5})=126.86^\circ & \textbf{and} \quad \theta_2=\sin^{-1}(-1)=\cos^{-1}(0)=-90^\circ \end{cases}$$





The Romin Robot. The Romin robot with three arms (the length of the first arm is invariant) and three degrees of freedom is located in $\mathbb{R}^3$. The angle θ_1 rotates around an axis perpendicular to the ground. Also, θ_2 (resp., θ_3) is the angle measured from the horizontal plane Q_1 (resp., Q_2) to the second arm (resp., third arm). For any point $(x, y, z) \in \mathbb{R}^3$ and also by considering $d^2 = x^2 + y^2$ and similar to the previous trends we obtain $F \subset \mathbb{C}[l_2, l_3, x, y, z, d][c_1, s_1, c_2, s_2, c_3, s_3]$ as the following: (For mor details see [6]) $F = \{s_1^2 + c_1^2 - 1, s_2^2 + c_2^2 - 1, s_3^2 + c_3^2 - 1, x + ds_1, y - dc_1, l_2c_2 + l_3c_3 - d, l_2s_2 + l_3s_3 - z\}$.



Since, $d^2 = x^2 + y^2$ so $d^2 - (x^2 + y^2) = 0$ and we compute a Gröbner system of F w.r.t. lexicographical ordering on variables and parameters with this assumption.

Parametric Constraints	Gröbner Bases
$[d, z, y, x, l_3, l_2], [\ ]$	$[c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1, c_3^2 + s_3^2 - 1]$
$[d, y, x, l_3, l_2], [z]$	$[c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1, -z]$
$[d, y, x, l_3, l_2 - z], [z]$	$[c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1, s_2z - z, c_2z, c_3^2 + s_3^2 - 1]$
$[d, y, x, l_3, l_2 + z], [z]$	$[c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1, -s_2z - z, -c_2z, c_3^2 + s_3^2 - 1]$
$[d, y, x, l_3], [l_2, l_2 - z, l_2 + z]$	$[c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1, l_2s_2 - z, l_2c_2, l_2^2 - z^2]$
$[d, y, x, l_3 - z, l_2], [z]$	$[c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1, s_3z - z, zc_3]$
$[d, y, x, l_3 + z, l_2], [z]$	$[c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1, -s_3z - z, -zc_3]$
$[d, y, x, l_2], [l_3, l_3 - z, l_3 + z]$	$[c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1, l_3s_3 - z, l_3c_3, -l_3^2 + z^2]$
$[d, z, y, x, l_2 - l_3], [l_3]$	$[c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1, l_3s_2 + l_3s_3, c_2l_3 + c_3l_3]$
$[d, z, y, x, l_2 + l_3], [l_3]$	$[c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1, -l_3s_2 + l_3s_3, -c_2l_3 + c_3l_3]$
$[d, z, y, x], [l_2, l_3, l_2 - l_3, l_2 + l_3]$	$[c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1, l_2s_2 + l_3s_3, c_2l_2 + c_3l_3, l_2^2 - l_3^2]$
$[d, y, x], [l_2, l_3, z]$	$[c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1, l_2s_2 + l_3s_3 - z, c_2l_2 + c_3l_3, -2l_2s_2z + l_2^2 - l_3^2 + z^2]$
$[z, -d^2 + x^2 + y^2, l_3, l_2], [d]$	$[ds_1 + x, c_1d - y, c_2^2 + s_2^2 - 1, -d]$
$[-d^2 + x^2 + y^2, l_3, l_2], [d, z]$	$[ds_1 + x, c_1d - y, c_2^2 + s_2^2 - 1, -z]$
$[x^2 + y^2 - d^2, l_3, l_2^2 - d^2 - z^2],$ $[d, l_2]$	$[ds_1 + x, c_1d - y, c_2^2 + s_2^2 - 1, l_2s_2 - z, c_2l_2 - d, c_3^2 + s_3^2 - 1]$
$[-d^2 + x^2 + y^2, l_3],$ $[d, l_2, -d^2 + l_2^2 - z^2]$	$[ds_1 + x, c_1d - y, c_2^2 + s_2^2 - 1, l_2s_2 - z, c_2l_2 - d, -d^2 + l_2^2 - z^2]$
$[x^2 + y^2 - d^2, l_3^2 - d^2 - z^2, l_2],$ $[d, l_3]$	$[ds_1 + x, c_1d - y, c_2^2 + s_2^2 - 1, l_3s_3 - z, c_3l_3 - d]$
$[-d^2 + x^2 + y^2, l_2],$ $[d, l_3, -d^2 + l_3^2 - z^2]$	$[ds_1 + x, c_1d - y, c_2^2 + s_2^2 - 1, l_3s_3 - z, c_3l_3 - d, d^2 - l_3^2 + z^2]$
$[d^2 + z^2, x^2 + y^2 - d^2, l_2 - l_3],$ $[d, l_3, z]$	$[ds_1 + x, c_1d - y, c_2^2 + s_2^2 - 1, l_3s_2 + l_3s_3 - z, c_2l_3 + c_3l_3 - d, -2c_2dl_3 - 2l_3s_2z, -4d^2l_3^2]$
$[d^2 + z^2, x^2 + y^2 - d^2, l_2 + l_3],$ $[d, l_3, z]$	$[ds_1 + x, c_1d - y, c_2^2 + s_2^2 - 1, -l_3s_2 + l_3s_3 - z, -c_2l_3 + c_3l_3 - d, 2c_2dl_3 + 2l_3s_2z, -4d^2l_3^2]$
$[d^2 + z^2, -d^2 + x^2 + y^2],$ $[d, l_2, l_3, z, l_2 - l_3, l_2 + l_3]$	$[ds_1 + x, c_1d - y, c_2^2 + s_2^2 - 1, l_2s_2 + l_3s_3 - z, c_2l_2 + c_3l_3 - d, -2c_2dl_2 - 2l_2s_2z + l_2^2 - l_3^2,$ $-4l_2^3s_2z + 4l_2l_3^2s_2z - 4d^2l_2^2 + l_2^4 - 2l_2^2l_3^2 + l_3^4]$
$[x^2 + y^2 - d^2],$ $[d, l_2, l_3, d^2 + z^2]$	$[ds_1 + x, c_1d - y, c_2^2 + s_2^2 - 1, l_2s_2 + l_3s_3 - z, c_2l_2 + l_3 - d, -2c_2dl_2 - 2l_2s_2z + d^2 +$ $l_2^2 - l_3^2 + z^2, 4d^2l_2^2s_2^2 + 4l_2^2s_2^2z^2 - 4d^2l_2s_2z - 4l_2^3s_2z + 4l_2l_3^2s_2z - 4l_2s_2z^3 + d^4 - 2d^2l_2^2 -$ $2d^2l_3^2 + 2d^2z^2 + l_2^4 - 2l_2^2l_3^2 + 2l_2^2z^2 + l_3^4 - 2l_3^2z^2 + z^4]$
$[z, l_3, l_2], [d, -d^2 + x^2 + y^2]$	$[ds_1 + x, c_1d - y, -d^2 + x^2 + y^2]$
$[l_3, l_2], [d, z, -d^2 + x^2 + y^2]$	$[ds_1 + x, c_1d - y, -d^2 + x^2 + y^2]$
$[l_3], [d, l_2, -d^2 + x^2 + y^2]$	$[ds_1 + x, c_1d - y, -d^2 + x^2 + y^2]$
$[\ ], [d, l_3, -d^2 + x^2 + y^2]$	$[ds_1 + x, c_1d - y, -d^2 + x^2 + y^2]$
Other caces (12 branches)	[1]

Now, we can analyze all possible parameter values to solve the inverse kinematics problem. Let us consider the 22nd row of the above table (the marked row) where $x^2 + y^2 - d^2 = 0$, $d \neq 0$, $l_2 \neq 0$, $l_3 \neq 0$, and $d^2 + z^2 \neq 0$. For example, if $(x, y, z) = (3, 4, 6) \in \mathbb{R}^3$ be the arm's end-effector position and $l2 = 3.5, l3 = 5.5$ be the lengths of two arms then these values satisfy the mentioned parametric constraints and so the specialization of G_{22} is $[5s_1+3, 5c_1-4, c_2^2+s_2^2-1, 5.5s_3+3.5s_2-6, 5.5c_3+3.5c_2-5, 43-35c_2-42s_2, 624-3612s_2+2989s_2^2]$ which discovers the following two solution sets:

$$\{s_3 = 0.9580021314 \quad c_3 = 0.2867610787 \quad s_2 = 0.2088537935, \quad c_2 = 0.9779468763, \quad s_1 = -0.6, \quad c_1 = 0.8\}$$
$$\{s_3 = 0.4548145601 \quad c_3 = 0.8905861642 \quad s_2 = 0.9995771198, \quad c_2 = 0.02907888479, \quad s_1 = -0.6, \quad c_1 = 0.8\}$$

Thus all status of rotation angles is as the following:

$$\begin{cases} \theta_3 = 73.33585444^\circ \quad \textbf{and} \quad \theta_2 = 12.05519009^\circ \quad \textbf{and} \quad \theta_1 = -36.86989764^\circ \\ \qquad\qquad\qquad\qquad \textbf{or} \\ \theta_3 = 27.05300333^\circ \quad \textbf{and} \quad \theta_2 = 88.33366771^\circ \quad \textbf{and} \quad \theta_1 = -36.86989764^\circ \end{cases}$$

2. Conclusion

Like solving the inverse kinematics problem in robots, many interesting applications can be examined using Gröbner systems with quite satisfactory results.

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