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A fixed point approach to stability of Jensen functional equations in probabilistic modular spaces

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Abstract. In this paper, we prove the generalized Hyers–Ulame stability of a generalized Jensen functional equation for mappings from linear spaces into β-homogeneous probabilistic modular spaces via fixed point method. Finally, we obtain some results for stability of the generalized Jensen functional equation in β-Banach spaces.

Keywords: stability, Jensen's functional equation, fixed point, probabilistic modular space.

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1. Introduction

The concept of stability for a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. Recall that the problem of stability of functional equations was motivated by a question of Ulam being asked in 1940 [14] and Hyers answer to it was published in [4]. Hyers's theorem was generalized by Aoki [1] for additive mappings and by Rassias [12] for linear mappings by considering an unbounded Cauchy difference. The result on the stability of the classical Jensen functional equation was first given by Kominek [8]. The author who presumably investigated the stability problem on a restricted domain for the first time was Skof [13]. The stability of the Jensen equation and its generalizations were studied by a number of mathematicians (cf., e.g., [5, 6, 9]). In this paper, by using some ideas of [6, 15], we investigate the generalized Hyers–Ulame stability of a generalized Jensen functional equation $f(rx+sy) = rg(x) + sh(x)$ for mappings from linear spaces into probabilistic modular spaces. The theory of modulars on linear spaces and the corresponding theory of modular linear spaces were founded by Nakano [10]. In [3], after introducing the probabilistic modular, authors then investigated some basic facts in such spaces and study linear operators defined between them.

Definition 1.1. Let $\mathcal{X}$ be an arbitrary vector space.
(a) A functional $\rho : \mathcal{X} \to [0,\infty]$ is called a modular if for arbitrary $x, y \in \mathcal{X}$,
(i) $\rho(x) = 0$ if and only if $x = 0$,

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(ii) $\rho(\alpha x) = \rho(x)$ for every scaler α with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,
(b) if (iii) is replaced by
(iii)$'$ $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,
then we say that ρ is a convex modular.

A modular ρ defines a corresponding modular space, i.e., the vector space $\mathcal{X}_\rho$ given by

$$\mathcal{X}_\rho = \{x \in \mathcal{X} : \quad \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

Let ρ be a convex modular, the modular space $\mathcal{X}_\rho$ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf\left\{\lambda > 0 \quad ; \quad \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

A function modular is said to satisfy the Δ_2–condition if there exists $\kappa > 0$ such that $\rho(2x) \leq \kappa\rho(x)$ for all $x \in \mathcal{X}_\rho$.

Definition 1.2. Let $\{x_n\}$ and x be in $\mathcal{X}_\rho$. Then
(i) the sequence $\{x_n\}$, with $x_n \in \mathcal{X}_\rho$, is ρ–convergent to x and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.
(ii) The sequence $\{x_n\}$, with $x_n \in \mathcal{X}_\rho$, is called ρ–Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
(iii) A subset $\mathcal{S}$ of $\mathcal{X}_\rho$ is called ρ–complete complete if and only if any ρ–Cauchy sequence is ρ–convergent to an element of $\mathcal{S}$.

The modular ρ has the Fatou property if and only if $\rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ–convergent to x.

Remark 1.3. Note that ρ is an increasing function. Suppose $0 < a < b$, then property (iii) of Definition 1.1 with $y = 0$ shows that $\rho(ax) = \rho\left(\frac{a}{b}bx\right) \leq \rho(bx)$ for all $x \in \mathcal{X}$. Moreover, if ρ is a convex modular on $\mathcal{X}$ and $|\alpha| \leq 1$, then $\rho(\alpha x) \leq \alpha\rho(x)$ and also $\rho(x) \leq \frac{1}{2}\rho(2x)$ for all $x \in \mathcal{X}$.

We follow the definition of probabilistic modular space briefly as given in [3]. In the following, Δ stands for the set of all non-decreasing functions $f : \mathbb{R} \to \mathbb{R}_0^+$ satisfying $\inf_{t\in\mathbb{R}} f(t) = 0$, and $\sup_{t\in\mathbb{R}} f(t) = 1$. We also denote the function min by $\wedge$.

Definition 1.4. A pair (X, μ) is called a probabilistic modular space ($\mathcal{PM}$-space) if X is a real vector space, μ is a mapping from X into Δ satisfying the following conditions:
(1) $\mu(x)(0) = 0$;
(2) $\mu(x)(t) = 1$ for all $t > 0$, if and only if $x = \theta$ (θ is the null vector in X);
(3) $\mu(-x)(t) = \mu(x)(t)$;
(4) $\mu(\alpha x + \beta y)(s + t) \geq \mu(x)(s) \wedge \mu(y)(t)$, for all $x, y \in X$, and $\alpha, \beta, s, t \in \mathbb{R}_0^+$, $\alpha + \beta = 1$.

For example, suppose that X is a real vector space and ρ is a modular on X. Define

$$\mu(x)(t) = \begin{cases} 0, & t \leq 0, \\ \frac{t}{t+\rho(x)}, & t > 0. \end{cases}$$

Then (X, μ) is a probabilistic modular space.

We say (X, μ) is β-homogeneous, where $\beta \in (0, 1]$ if,

$$\mu(\alpha x)(t) = \mu(x)\left(\frac{t}{|\alpha|^\beta}\right)$$

for every $x \in X$, $t > 0$, and $\alpha \in \mathbb{R} \setminus \{0\}$.

DEFINITION 1.5. Let (X,μ) be a $\mathcal{PM}$-space, $\{x_n\}$ be a sequence in X and $x \in X$. Then
(i) the sequence $\{x_n\}$, with $x_n \in (X,\mu)$, is μ–convergent to x and write $x_n \xrightarrow{\mu} x$, if for every $t>0$ and $\lambda \in (0,1)$, there exists a positive integer n_0 such that $\mu(x_n - x)(t) > 1-\lambda$ for all $n \geq n_0$.
(ii) the sequence $\{x_n\}$, with $x_n \in (X,\mu)$, is μ–Cauchy, if for every $t>0$ and $\lambda \in (0,1)$, there exists a positive integer n_0 such that $\mu(x_n - x_m)(t) > 1-\lambda$ for all $m,n \geq n_0$.

By [3], every μ-convergent sequence in a $\mathcal{PM}$-space is a μ-Cauchy sequence. If each μ-Cauchy sequence is μ-convergent in a $\mathcal{PM}$-space (X,μ), then (X,μ) is called a *μ-complete* $\mathcal{PM}$-space.
A $\mathcal{PM}$-space (X,μ) possesses Fatou property if for any sequence $\{x_n\}$ of X μ-converging to x, we have

$$\mu(x)(t) \geq \limsup_{n\geq 1} \mu(x_n)(t)$$

for each $t>0$.

REMARK 1.6. Note that for any $x \in X$, $\mu(x)(.)$ is an increasing function, Since $\mu(x) \in \Delta$. Moreover, if μ is a β-homogeneous probabilistic modular on X and $x,y \in X$, then property (4) of Definition 1.4 shows that

$$\mu(x+y)\left(2^{\beta}(s+t)\right) = \mu\left(\frac{1}{2}x + \frac{1}{2}y\right)(s+t) \geq \mu(x)(s) \wedge \mu(y)(t).$$

For more details about the $\mathcal{PM}$-space, the readers refer to [11].

Our aim is based on the fixed point approach:

THEOREM 1.7 ([7]). *Let X_ρ be a modular space such that satisfies the Fatou property. Let $\mathcal{C}$ be a ρ-complete nonempty subset of X_ρ and let $T:\mathcal{C}\to\mathcal{C}$ be quasicontraction, that is, there exists $K<1$ such that*

$$\rho(T(x)-T(y)) \leq K\max\{\rho(x-y), \rho(x-T(x)), \rho(y-T(y)), \rho(x-T(y)), \rho(y-T(x))\}.$$

Let $x \in \mathcal{C}$ such that

$$\delta_\rho(x) := \sup\{\rho(T^n(x)-T^m(x)) : m,n \in \mathbb{N}\} < \infty.$$

Then $T^n(x)$ ρ-converges to $\omega \in \mathcal{C}$. Moreover, if $\rho(\omega - T(\omega)) < \infty$ and $\rho(x-T(\omega)) < \infty$, then, the ρ-limit of $T^n(x)$ is a fixed point of T. Furthermore, if ω^* is any fixed point of T in $\mathcal{C}$ such that $\rho(\omega - \omega*) < \infty$, then one has $\omega = \omega*$.

Throughout this paper, we assume that μ is a probabilistic modular on X with the Fatou property(in the probabilistic modular sense) and (X,μ) is a μ-complete β-homogeneous $\mathcal{PM}$-space with $\beta \in (0,1]$.

2. Main results

Now, we assume that r,s constant positive integer numbers. we are ready to prove stability the functional equation $f(rx+sy) = rg(x) + sh(y)$.

THEOREM 2.1. *Let $f,g,h : E \to (X,\mu)$ be mappings with $f(0)=g(0)=h(0)=0$ satisfying*

(1) $$\mu(f(rx+sy) - rg(x) - sh(y))(t) \geq \phi(x,y)(t)$$

for all $x, y \in E$*, where* $\phi : E \times E \to \Delta$ *is given function. If there exists* $0 < L < \frac{1}{2^\beta}$ *such that*

$$\phi(2x, 2x)(2^\beta Lt) \geq \phi(x, x)(t)$$

and has the property

$$\lim_{n\to\infty} \phi(2^n x, 2^n y)(2^{\beta n} t) = 1 \tag{2}$$

for all $x, y \in E$*. Then there exists a unique additive mapping* $\mathcal{A} : E \to (X, \mu)$ *such that*

$$\begin{aligned} &\mu(f(x) - \mathcal{A}(x))(t) \geq \Phi(x, x)(t) \\ &\mu(g(x) - \mathcal{A}(x))\left(\frac{2^{\beta+1}t}{r^\beta}\right) \geq \phi(x, 0)(t) \wedge \Phi(rx, rx)(t) \\ &\mu(h(x) - \mathcal{A}(x))\left(\frac{2^{\beta+1}t}{s^\beta}\right) \geq \phi(0, x)(t) \wedge \Phi(sx, sx)(t) \end{aligned}$$

for all $x \in E$*, where*

$$\Phi(x, y)(t) := \phi\left(\frac{x}{r}, \frac{y}{s}\right)\left(\tfrac{(1-2^\beta L)t}{2^\beta(2^{\beta+1}+1)}\right) \wedge \phi\left(\frac{x}{r}, 0\right)\left(\tfrac{(1-2^\beta L)t}{2^\beta(2^{\beta+1}+1)}\right) \wedge \phi\left(0, \frac{y}{s}\right)\left(\tfrac{(1-2^\beta L)t}{2^\beta(2^{\beta+1}+1)}\right).$$

Proof. Letting $y = 0$ in (1) we get

$$\mu(f(rx) - rg(x))(t) \geq \phi(x, 0)(t)$$

for all $x \in E$. Letting $x = 0$ in (1) we get

$$\mu(f(sy) - sh(y))(t) \geq \phi(0, y)(t)$$

for all $y \in E$. Then

$$\begin{aligned} &\mu(f(rx + sy) - f(rx) - f(sy))\left(2^\beta\left(2^{\beta+1} + 1\right)t\right) \\ &\geq \mu\left(f(rx + sy) - rg(x) - sh(y)\right)(t) \wedge \mu\left(rg(x) - f(rx) - f(sy) + sh(y)\right)\left(2^{\beta+1}t\right) \\ &\geq \mu\left(f(rx + sy) - rg(x) - sh(y)\right)(t) \wedge \mu(f(rx) - rg(x))(t) \wedge \mu(f(sy) - sh(y))(t) \\ &\geq \phi(x, y)(t) \wedge \phi(x, 0)(t) \wedge \phi(0, y)(t). \end{aligned}$$

Replacing x by x/r, y by y/r and t by $t/2^\beta(2^{\beta+1} + 1)$ in the above inequality, we obtain

$$\begin{aligned} &\mu(f(x + y) - f(x) - f(y))(t) \\ &\geq \phi\left(\frac{x}{r}, \frac{y}{s}\right)\left(\frac{t}{2^\beta(2^{\beta+1} + 1)}\right) \wedge \phi\left(\frac{x}{r}, 0\right)\left(\frac{t}{2^\beta(2^{\beta+1} + 1)}\right) \wedge \phi\left(0, \frac{y}{s}\right)\left(\frac{t}{2^\beta(2^{\beta+1} + 1)}\right) \end{aligned}$$

for all $x, y \in E$. By Theorem 1.7 and [15, Theorem 2.1], there exists a unique additive mapping $\mathcal{A} : E \to (X, \mu)$ given by $\mathcal{A}(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ such that

$$\mu(f(x) - \mathcal{A}(x))(t) \geq \Phi(x, x)(t) \tag{3}$$

for all $x \in E$. Since $\mathcal{A}$ is a additive, we have $\mathcal{A}(qx) = q\mathcal{A}(x)$ for all rational numbers q and $x \in E$. It follows from inequalities (1) and (3) that

$$\begin{aligned} \mu(g(x) - \mathcal{A}(x))\left(\frac{2^{\beta+1}t}{r^\beta}\right) &\geq \mu\left(g(x) - \frac{1}{r}f(rx)\right)\left(\frac{t}{r^\beta}\right) \wedge \mu\left(\frac{1}{r}f(rx) - \mathcal{A}(x)\right)\left(\frac{t}{r^\beta}\right) \\ &\geq \mu(rg(x) - f(rx))(t) \wedge \mu(f(rx) - \mathcal{A}(rx))(t) \\ &\geq \phi(x, 0)(t) \wedge \Phi(rx, rx)(t) \end{aligned}$$

for all $x \in E$. Similarly, we obtain the following inequality

$$\mu(g(x) - \mathcal{A}(x))\left(\frac{2^{\beta+1}t}{s^{\beta}}\right) \geq \phi(0,x)(t) \wedge \Phi(sx,sx)(t)$$

for all $x \in E$. □

Next, we give an example of the generalized Hyers–Ulam stability of the generalized Jensen functional equation in β-Banach space. We firstly introduce some useful concepts: We fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A real-valued function $\| \, . \, \|_\beta$ is called a *β-norm* on X if and only if it satisfies

($\beta N1$) $\|x\|_\beta = 0$ if and only if $x = 0$;

($\beta N2$) $\|\lambda x\|_\beta = |\lambda|^\beta . \, \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$;

($\beta N3$) $\|x + y\|_\beta \leq \|x\|_\beta + \|y\|_\beta$ for all $x, y \in X$.

The pair $(X, \| \, . \, \|_\beta)$ is called a *β-normed space* (see [2]). A β-Banach space is a complete β-normed space.

Example 2.2. Let E is a linear space and X is a β-Banach space. Define

$$\mu(x)(t) = \begin{cases} 0, & t \leq 0, \\ \frac{t}{t+\|x\|_\beta}, & t > 0, \end{cases}$$

for all $x \in X$ and $t \in \mathbb{R}$. Then (X, μ) is a μ-complete β-homogeneous probabilistic modular space. Moreover, let f, g, h, ϕ, Φ and L be as in Theorem 2.1. Then there exists a unique additive function $\mathcal{A} : E \to (X, \mu)$ satisfying

$$\frac{t}{t + \|j(x) - f(x)\|_\beta} \geq \Phi(x,x)(t),$$

$$\frac{2^{\beta+1}t}{2^{\beta+1}t + s^\beta\|g(x) - \mathcal{A}(x)\|_\beta} \geq \phi(x,0)(t) \wedge \Phi(rx,rx)(t),$$

$$\frac{2^{\beta+1}t}{2^{\beta+1}t + s^\beta\|h(x) - \mathcal{A}(x)\|_\beta} \geq \phi(0,x)(t) \wedge \Phi(sx,sx)(t),$$

for all $x \in E$.

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