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Fuzzy n-fold obstinate filters on hoop algebras

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Abstract. In this paper, we introduce the concepts of fuzzy n-fold obstinate filter on hoop algebras and study some properties of them. We define and investigate fuzzy n-fold implicative filter on hoop algebra. Also, the relation between fuzzy n-fold obstinate filter and n-fold obstinate hoop algebra are investigated. We obtain some condition equivalent with fuzzy n-fold implicative filter and show that every fuzzy n-fold obstinate filter is a fuzzy n-fold implicative filter.

Keywords: Hoop algebra, fuzzy filter, fuzzy n-fold obstinate filter.

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1. Introduction

Naturally ordered commutative residuated integral monoids (hoop algebra) introduced by B. Bosbach in [5, 6]. It is well known that in various logical systems, filter play a fundamental role, filters correspond to sets of provable formulas closed with respect to Modus Ponnen. In last years, hoop theorey was enriched with deep structure theorems. Several researchers investigated the theory of filter and n-fold filters on hoop algebra, in 2017, C. luo, X.Li, P. He, study n-fold filters theory on hoop algebras and several characterizations of n-fold filters, implicative and positive implicative are investigated. They show that if A is a n-fold (positive)implicative hoop algebra, then every filter of A is an n-fold (positive)implicative filter and A/F is a n-fold (positive)implicative hoop. Furthermore, the relation between these n-fold filters on hoop algebras is established. n-fold obstinate filter on hoop algebra defined and some relations between these filter and (positive) implicative filters, maximal filters, prime filters , fantastic filters and perfect filters on hoop algebras were investigated [4]. In the fuzzy approach, fuzzification ideas have been applied to some fuzzy logical algebras. In [2], fuzzy filters on pseudo hoop algebras were studied. In particular, several types of fuzzy filters such as fuzzy implicative filters, fuzzy positive implicative filters, fuzzy Boolean filters and fuzzy fantastic filters were introduced. This paper continues the study of fuzzy n-fold filters on hoop algebras. Some basic concepts and properties are recalled, and some new notions about the thresholds

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are introduced to represent fuzzy n-fold filters like fuzzy n-fold obstinate filters which are convenient to study the properties of fuzzy n-fold filters. In Section 3, the relation between fuzzy n-fold obstinate filters and fuzzy n-fold filters on hoop algebras is established and we obtained the condition equivalent to the fuzzy n-fold obstinate filter. We show that extension theorem of fuzzy n-fold obstinate filter on hoop algebra is established and the preimage of a fuzzy n-fold obstinate filter μ under f is a fuzzy n-fold obstinate filter on hoop algebra. In special case, we prove that if μ_α is a n-fold obstinate filter, then μ is a fuzzy n-fold obstinate filter of A. Also if μ is a fuzzy n-fold obstinate filter of A, then level subset μ_α is an n-fold obstinate filter of A. Also, we introduce the concept of fuzzy n-fold implicative filters and we prove some related results and obtained some conditions equivalent with fuzzy n-fold implicative filters and study relation between fuzzy n-fold obstinate filter and fuzzy n-fold implicative filter. Finally, we show that a quotient n-fold obstinate hoop algebra with respect to fuzzy filter is a n-fold obstinate hoop algebra.

2. Preliminaries

In this section, we collect some definitions and results which will be used, not cite them every time they are used.

Definition 2.1. [1] A *hoop algebra* or *hoop* is an algebra $(A, \odot, \to, 1)$ of type $(2, 2, 0)$ such that, for all $x, y, z \in A$:
(HP1) $(A, \odot, 1)$ is a commutative monoid,
(HP2) $x \to x = 1$,
(HP3) $(x \odot y) \to z = x \to (y \to z)$,
(HP4) $x \odot (x \to y) = y \odot (y \to x)$.

On hoop A, we define $x \leq y$ if and only if $x \to y = 1$. It is easy to see that $\leq$ is a partial order relation on A. A hoop A is *bounded* if there is an element $0 \in A$ such that $0 \leq x$, for all $x \in A$. In this case, we define a negation "$'$" on A by, $x' = x \to 0$, for all $x \in A$. We denoted $(x \to (...(x \to (x \to y)))...)$ by $x^n \to y$, for all $x, y \in A$.

Proposition 2.2. [5,6] *Let A be a bounded hoop. Then the following properties hold, for all $x, y, z \in A$:*
(i) $(A, \leq)$ *is a meet-semilattice with* $x \wedge y = x \odot (x \to y)$,
(ii) $x \odot (x \to y) \leq y$ *and* $x \odot (y \to z) \leq y \to x \odot z$,
(iii) $x \to 1 = 1,\ 1 \to x = x,\ 0 \to x = 1,\ x \odot y \leq x, y,\ 0 \leq y$,
(iv) *if* $x \leq y$, *then* $y' \leq x'$, $z \to x \leq z \to y$ *and* $y \to z \leq x \to z$,
(v) $x \leq (x \to y) \to y,\ x \leq y \to x$.

Definition 2.3. [2,4,8] Let F be a non-empty subset of A such that $1 \in F$. Then for any $x, y, z \in A$:
(i) F is called a *filter* if $x, x \to y \in F$, then $y \in F$.
(ii) F is called an *n-fold implicative filter* of A, if $x \to ((y^n \to z) \to y) \in F$ and $x \in F$, then $y \in F$.
(iii) F is called an *n-fold obstinate filter* of A, if F is a proper filter and for any $x, y \notin F$, $x^n \to y \in F$ and $y^n \to x \in F$.
(iv) A bounded hoop A is called a *n-fold implicative bounded hoop* if $(x^n \to 0) \to x = x$ for any $x \in A$.
(v) A bounded hoop A is called an *n-fold obstinate hoop* if $x^n = 0$ for any $x \neq 1$.
(vi) A fuzzy subset μ of A is called *fuzzy filter* if $\mu(x) \leq \mu(1)$ and $\mu(x \to y) \wedge \mu(x) \leq \mu(y)$.

If A is a bounded hoop, then a filter is proper if and only if it is not containing 0.

DEFINITION 2.4. [7] Let A and B be two bounded hoops. A map $f : A \to B$ is called a *hoop homomorphism* if and only if for all $x, y \in A$, $f(0) = 0$, $f(1) = 1$, $f(x \odot y) = f(x) \odot f(y)$ and $f(x \to y) = f(x) \to f(y)$.

If f is a hoop homomorphism, then $f(x^n \to y) = f(x^n) \to f(y)$.

DEFINITION 2.5. [2] Let μ be a fuzzy filter of A and $A/\mu = \{\mu^x \mid x \in A\}$. For all $\mu^x, \mu^y \in A/\mu$, define $\mu^x \to \mu^y = \mu^{x \to y}$, $\mu^x \odot \mu^y = \mu^{x \odot y}$ and $\mu^x : A \to [0, 1]$ which is defined by $\mu^x(y) = \mu(x \to y) \wedge \mu(y \to x)$.

PROPOSITION 2.6. [2, 8] *(i) μ is a fuzzy implicative filter of A if and only if $\mu_\alpha \neq \emptyset$ is an implicative filter for any $\alpha \in [0, 1]$.*
(ii) A is a n-fold implicative bounded hoop if and only if $\{1\}$ is an n-fold implicative filter.
(iii) Let μ be a fuzzy filter of A and $x \leq y$. Then $\mu(x) \leq \mu(y)$.

Notation: From now one, we let $(A, \odot, \to, 1)$ or A is a hoop, unless otherwise state.

3. A new fuzzy filter on hoop algebra

In this section, we introduce the notion of fuzzy n-fold obstinate filter on hoop algebra and investigate some properties of them.

DEFINITION 3.1. A fuzzy filter μ of A is a fuzzy n-fold obstinate filter if for any $x, y \in A$:

$$(1 - \mu(x)) \wedge (1 - \mu(y)) \leq \mu(x^n \to y) \wedge \mu(y^n \to x).$$

In particular, fuzzy 1-fold obstinate filters are fuzzy obstinate filters.
In the following example, we show that any fuzzy filter may not be fuzzy n-fold obstinate filter.

EXAMPLE 3.2. Let $A = \{0, a, b, c, d, 1\}$ and define the operations $\odot$ and $\to$ on A as follows:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	d	0	d	a
b	0	d	c	c	0	b
c	0	0	c	c	0	c
d	0	d	0	0	0	d
1	0	a	b	c	d	1

Then $(A, \odot, \to, 1, 0)$ is a bounded hoop algebra.
(i) It it is easy to see that $\mu(x) = \frac{1}{2}$ for any $x \in A$, is a fuzzy n-fold obstinate filter.
(ii) Let $\mu(0) = \mu(a) = \mu(c) = \mu(b) = \mu(d) = \frac{1}{3}$, $\mu(1) = \frac{2}{3}$. Then $\mu(x)$ is a fuzzy filter, but is not a fuzzy 2-fold obstinate filter. Because $(1-\mu(b)) \wedge (1-\mu(0)) = \frac{2}{3} = (1-\frac{1}{3}) \wedge (1-\frac{1}{3}) \nleq \mu(a) \wedge \mu(1) = \mu(b^2 \to 0) \wedge \mu(0^2 \to b)$.

THEOREM 3.3. *Let μ be a fuzzy filter of A. Then μ is a fuzzy n-fold obstinate filter if and only if for any $x \in A$, $1 - \mu(x) \leq \mu(x^n)'$.*

PROOF. Let for any $x \in A$, $1 - \mu(x) \leq \mu(x^n)'$. Then $1 - \mu(y) \leq \mu(y^n)'$. Hence $1 - \mu(x) \leq \mu(x^n \to 0)$ and $1 - \mu(y) \leq \mu(y^n \to 0)$. By Proposition 2.2(iv), $x^n \to 0 \leq x^n \to y$ and $y^n \to 0 \leq y^n \to x$. Since μ is a fuzzy filter, by Proposition 2.2(iii),

$\mu(x^n \to 0) \le \mu(x^n \to y)$ and $\mu(y^n \to 0) \le \mu(y^n \to x)$. So, $1 - \mu(x) \le \mu(x^n \to y)$ and $1 - \mu(y) \le \mu(y^n \to x)$. Hence

$$(1 - \mu(x)) \wedge (1 - \mu(y)) \le \mu(x^n \to y) \wedge \mu(y^n \to x).$$

Thus μ is a fuzzy n-fold obstinate filter of A.
Conversely, let μ is a fuzzy n-fold obstinate filter of A. Then for any $x, y \in A$, $(1-\mu(x)) \wedge (1 - \mu(y)) \le \mu(x^n \to y) \wedge \mu(y^n \to x)$. If $y = 0$, then $1 - \mu(x) \le \mu(x^n \to 0)$. Therefore, $1 - \mu(x) \le \mu(x^n)'$. □

THEOREM 3.4. *(Extension theorem of fuzzy n-fold obstinate filters) Let μ, λ be two fuzzy filters of A such that $\mu \subseteq \lambda$. If μ is a fuzzy n-fold obstinate filter, then λ is a fuzzy n-fold obstinate filter of A.*

THEOREM 3.5. *If μ is a fuzzy n-fold obstinate filter of A, then μ is a fuzzy $(n+1)$-fold obstinate filter of A.*

By finite induction, we can prove that every fuzzy n-fold obstinate filter is a fuzzy $(n + k)$-fold obstinate filter for any integer $k \ge 0$.
The following example shows that any fuzzy $(n + 1)$- fold obstinate filter may not be a fuzzy n-fold obstinate filter of A.

EXAMPLE 3.6. Let $(A = \{0, a, b, 1\}, \le)$ be a chain. Define the operations $\odot$ and $\to$ on A as follows:

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Then $(A, \odot, \to, 1, 0)$ is a bounded hoop algebra. Define a fuzzy set on A by $\mu(b) = \frac{8}{10} = \mu(1)$, $\mu(0) = \frac{3}{10} = \mu(a)$. Using Theorem 3.3, for $n = 2$, it is easy to see that μ is a fuzzy 2-fold obstinate filter of A but is not a fuzzy 1-fold obstinate filter of A. Because $\frac{7}{10} = 1 - \mu(a)) \nleq \mu(a') = \mu(a) = \frac{3}{10}$.

PROPOSITION 3.7. *Any non empty subset F of A is an n-fold obstinate filter if and only if the characteristic function χ_F is a fuzzy n-fold obstinate filter.*

THEOREM 3.8. (i) *Let μ be a fuzzy n-fold obstinate filter of A for any $\alpha \in [0, \frac{1}{2}]$. Then level subset $\mu_\alpha = \{x \in A \mid \mu(x) \ge \alpha\}$ is an n-fold obstinate filter of A.*
(ii) *If $\mu_\alpha \neq \emptyset$ is an n-fold obstinate filter and $\alpha \in (\frac{1}{2}, 1]$, then μ is a fuzzy n-fold obstinate filter of A.*

COROLLARY 3.9. *Let μ be a fuzzy n-fold obstinate filter of A and for any $\mu(1) \in [0, \frac{1}{2}]$. Then level subset $I = \{x \in A \mid \mu(x) = \{1\}\}$ is an n-fold obstinate filter of A.*

DEFINITION 3.10. Let A and B be two hoop algebras and μ be a fuzzy subset of A and λ a fuzzy subset of B such that $f : A \to B$ be a hoop homomorphism. Then image μ under f denoted by $f(\mu)$ is a fuzzy set of B that for any $y \in B$:
$f(\mu)(y) = sup_{x \in f^{-1}(y)} \mu(x)$ if $f^{-1}(y) \neq \emptyset$ and $f(\mu)(y) = 0$ if $f^{-1}(y) = \emptyset$.
The preimage of λ under f denoted by $f^{-1}(\lambda)$ is a fuzzy set of A denoted by for any $x \in A$, $f^{-1}(\lambda)(x) = \lambda(f(x))$.

DEFINITION 3.11. Fuzzy subset μ of A has sup property if for any nonempty subset Y of A, there exists $y_0 \in Y$ such that $\mu(y_0) = sup_{y \in Y} \mu(y)$.

EXAMPLE 3.12. In Example 3.6, let $Y_1 = \{0, a\}$, $a \in Y_1$ such that $\mu(a) = sup_{y \in Y_1} \mu(y)$. Also $Y_2 = \{0, b\}$, $b \in Y_2$ such that $\mu(b) = sup_{y \in Y_2} \mu(y)$.

PROPOSITION 3.13. (i) *Let $f : A \to B$ be an onto hoop homomorphism algebra. Then preimage of a fuzzy n-fold obstinate filter μ under f is a fuzzy n-fold obstinate filter of A.*
(ii) *Let $f : A \to B$ be an onto hoop homomorphism algebra and μ be a fuzzy n-fold obstinate filter of A with sup property. Then $f(\mu)$ is a fuzzy n-fold obstinate filter of B.*

PROPOSITION 3.14. *Let A be an n-fold obstinate hoop algebra and μ be a fuzzy filter.*
(i) *If for any $x \in A$, $\mu(x) \leq \frac{1}{2} \leq \mu(1)$, then any fuzzy filter is a fuzzy n-fold obstinate filter of A.*
(ii) *A/μ is an n-fold obstinate hoop algebra.*

4. Fuzzy n-fold implicative filter

In this section, we introduce the notion of fuzzy n-fold implicative filter on hoop algebra and investigate some properties of them. We obtained some conditions are equivalent for fuzzy n-fold implicative filters.

DEFINITION 4.1. fuzzy subset μ of A is called *fuzzy n-fold implicative filter* of A if $\mu(x) \leq \mu(1)$ and

$$\mu(x \to ((y^n \to z) \to y)) \wedge \mu(x) \leq \mu(y).$$

EXAMPLE 4.2. In Example 3.6, let $\mu(0) = \mu(a) = \mu(b) = \mu(c) = \mu(d) = \frac{1}{3}, \mu(1) = \frac{1}{2}$. Then μ is a fuzzy 2-fold implicative filter

PROPOSITION 4.3. *Every fuzzy n-fold implicative filter of A is a fuzzy filter of A.*

PROOF. Let μ be a fuzzy n-fold implicative filter of A. Then $\mu(x) \leq \mu(1)$. If $z = 1$, then $\mu(x \to ((y^n \to 1) \to y)) \wedge \mu(x) \leq \mu(y)$. Then $\mu(x \to y) \wedge \mu(x) \leq \mu(y)$. □

THEOREM 4.4. *μ is a fuzzy n-fold implicative filter of A if and only if $\mu_\alpha \neq \emptyset$ is an n-fold implicative filter of A, for any $\alpha \in [0, 1]$.*

THEOREM 4.5. *Let μ be a fuzzy filter of A. Then for any $x, y \in A$, the following conditions are equivalent:*
(i) *μ is a fuzzy n-fold implicative filter of A,*
(ii) $\mu((x^n \to y) \to x) \leq \mu(x)$,
(iii) $\mu(((x^n \to y) \to x) \to x) = \mu(1)$,
(iv) $\mu(((x^n)' \to x) \to x) = \mu(1)$.

PROPOSITION 4.6. *Let fuzzy set μ of A is defined by*

$$\mu(x) = \begin{cases} 0 & x \neq 1 \\ \alpha & x = 1 \end{cases}$$

for $\alpha \in (0, 1]$. Then the following are equivalent:
(i) *A is an n-fold implicative hoop algebra,*
(ii) *Any fuzzy filter is a fuzzy n-fold implicative filter of A,*
(iii) *μ is a fuzzy n-fold implicative filter of A.*

THEOREM 4.7. *μ is a fuzzy n-fold implicative filter if and only if $\mu((x^n \to 0) \to x) \leq \mu(x)$, for any $x \in A$.*

Theorem 4.8. *Let μ be a fuzzy n-fold implicative filter of A. Then $\mu((x^n \to y) \to y) = \mu((y^n \to x) \to x)$, for any $x, y \in A$.*

Proposition 4.9. *If μ is a fuzzy n-fold implicative filter of A, then μ is a fuzzy $(n+1)$-fold implicative filter of A.*

The following example shows that the converse of Proposition 4.9, is not true in general.

Example 4.10. Let $(A = \{0, a, b, 1\}, \leq)$. Define the operations $\odot$ and $\to$ on A as follows:

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

Then $(A, \odot, \to, 1, 0)$ is a bounded hoop algebra. Define a fuzzy set in A by $\mu(0) = \mu(a) = \mu(b) = \frac{1}{4}$, $\mu(1) = \frac{3}{4}$. Using Theorem 4.7, for $n = 3$, it is easy to see that μ is a fuzzy 3-fold implicative filter of A but is not a fuzzy 2-fold implicative filter of A. Because $\mu((b^2 \to 0) \to b) = \mu(1) \nleq \mu(b)$.

Proposition 4.11. *Every fuzzy n-fold obstinate filter μ of A such that $\mu(x) < \frac{1}{2}$ is a fuzzy n-fold implicative filter, for any $x \in A$.*

5. Conclusion

In this paper, we investigated fuzzy n-fold obstinate filter and fuzzy n-fold implicative filter on hoop algebras and studied properties of them. Then we obtained relation between fuzzy n-fold obstinate filter with fuzzy n-fold implicative filters. Hence A is an n-fold implicative hoop algebra if and only if any fuzzy filter is a fuzzy n-fold implicative filter if and only if μ is a fuzzy n-fold implicative filter.

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